### Lesson Objectives

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<td>• Solve exponential equations and inequalities using common logarithms.</td>
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*Pacing suggestions for the entire year can be found on pages T20–T21.*
### Chapter Resource Manager

#### CHAPTER 10 RESOURCE MASTERS

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*Key to Abbreviations: GCS = Graphing Calculator and Spreadsheet Masters, SC = School-to-Career Masters, SM = Science and Mathematics Lab Manual*
10-1 Exponential Functions

Examine the list of characteristics for an exponential function on page 524. The first characteristic states that an exponential function is continuous and one-to-one. The term continuous means that the function can be traced without lifting your pencil. The term one-to-one means that a horizontal line passing through the graph will intersect no more than one point on the graph. This characteristic is important for the development of the logarithmic function in Lesson 10-2, since only one-to-one functions can have inverses. The second characteristic listed is that the domain of the function is the set of all real numbers. This property is important because it means that \( \frac{3}{5} \) has meaning, since \( \sqrt{5} \) is a real number and part of the domain of \( y = 3^x \). The third and fourth characteristics of an exponential function are related. The \( x \)-axis is a horizontal asymptote of the graph of an exponential function. This means that the graph of this function approaches the horizontal line \( x = 0 \), getting closer and closer to this line but never crossing it. This restricts the graph of an exponential function to either Quadrants I and II, when \( a \) is positive, or to Quadrants III and IV, when \( a \) is negative. In terms of the range of the function, this means that when \( a \) is positive, all \( y \) values of the function will be positive, and when \( a \) is negative, all \( y \) values of the function will be negative. These two properties will also be important when considering the inverse of the exponential function. The last two properties are useful for graphing and writing exponential functions.

10-2 Logarithms and Logarithmic Functions

In the equation \( y = \log_b x \), \( y \) is referred to as the logarithm, \( b \) is the base, and \( x \) is sometimes referred to as the argument. The definition of a logarithm given on page 532 indicates that a logarithm is an exponent. When solving logarithmic equations and inequalities, it is important to remember that a defining characteristic of a logarithmic function is that its domain is the set of all positive numbers. This means that the logarithm of 0 or of a negative number for any base is undefined. It is very important to check possible solutions to logarithmic equations in the original equation, to be sure that they would not result in taking the logarithm of 0 or a negative number. For logarithmic inequalities, this fact will exclude not just one value from the solution set, but a range of values. In Example 8 on page 534, since the original inequality asks for the values \( \log_{10} (3x - 4) \) and \( \log_{10} (x + 6) \), we must solve two inequalities, \( 3x - 4 \leq 0 \) and \( x + 6 \leq 0 \), to find what values must be excluded from
the solution set we found using the Property of Inequality for Logarithmic Functions. Excluding the values such that \( x \leq \frac{4}{3} \) and \( x \leq -6 \), the solution set is all \( x \) such that the following three inequalities are all satisfied: \( x > \frac{4}{3} \), \( x > -6 \), and \( x < 5 \). To simplify this compound inequality, sketch all three inequalities, as shown below, and find where all three intersect.

\[
\begin{align*}
&\ x > \frac{4}{3} \\
&\ x > -6 \\
&\ x < 5
\end{align*}
\]

The final number line shows that the solution set is the compound inequality \( \frac{4}{3} < x < 5 \).

### 10-3 Properties of Logarithms

The word logarithm is actually a contraction of “logical arithmetic.” Logarithms were invented to make computation easier. Using logarithms, multiplication changes to addition, according to the Product Property of Logarithms, and division changes to subtraction, according to the Quotient Property of Logarithms. This is illustrated in Examples 1, 2, and 4 of Lesson 10-3. In these examples, students are given the approximate value of specific logarithms. Before the invention of the scientific calculator, these values took a good deal of time to compute. Rather than use the same arduous process to compute each and every logarithm one encountered, the properties of logarithms allowed the use of a relative few logarithmic values to compute others.

### 10-4 Common Logarithms

Before the invention of the scientific calculator, the appendices of algebra texts contained extensive tables of common logarithms of numbers. In order to read these tables, you had to understand the parts of a logarithm. Every logarithm has two parts, the characteristic and the mantissa. A mantissa is the logarithm of a number between 1 and 10. When the original number is expressed in scientific notation, the characteristic is the power of 10.

\[
\begin{align*}
645,000 &= 6.45 \cdot 10^5 \\
\log 645,000 &= \log (6.45 \cdot 10^5) \\
&= \log 6.45 + \log 10^5 \\
&= \log 6.45 + 5 \\
&= 0.8096 + 5 \\
&= 5.8096
\end{align*}
\]

\( \log 645,000 \approx 5.8096 \)

Simplify.

### 10-5 Base e and Natural Logarithms

Exponentiation, which is the inverse operation of taking a logarithm, is sometimes referred to as finding the antilogarithm. That is, if \( \log x = a \) then \( x = \text{antilog} \ a \). Since antilogarithms mean the same operation as exponentiation, it follows that to find the antilogarithm of a common logarithm, you would use \( \text{2nd} \ 10^x \) on a graphing calculator. To find the antilogarithm of a natural logarithm, \( \text{antiln} \ a \), you would use \( \text{2nd} \ e^x \).

### 10-6 Exponential Growth and Decay

It is important to note that the variable \( r \) in the exponential decay formula \( y = a(1 - r)^t \) and the variable \( k \) in the alternate exponential decay formula \( y = ae^{-kt} \) are not equivalent. In a problem where a decay factor is given or asked for, the formula \( y = a(1 - r)^t \) should be used and not the formula \( y = ae^{-kt} \). The same is true of the exponential growth formulas \( y = a(1 + r)^t \) and \( y = ae^{kt} \).
<table>
<thead>
<tr>
<th>Type</th>
<th>Student Edition</th>
<th>Teacher Resources</th>
<th>Technology/Internet</th>
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Key to Abbreviations: TWE = Teacher Wraparound Edition; CRM = Chapter Resource Masters

Additional Intervention Resources
The Princeton Review’s *Cracking the SAT & PSAT*
The Princeton Review’s *Cracking the ACT*
ALEKS

TestCheck and Worksheet Builder
This networkable software has three modules for intervention and assessment flexibility:
- **Worksheet Builder** to make worksheet and tests
- **Student Module** to take tests on screen (optional)
- **Management System** to keep student records (optional)

Special banks are included for SAT, ACT, TIMSS, NAEP, and End-of-Course tests.
ALEKS is an online mathematics learning system that adapts assessment and tutoring to the student’s needs. Subscribe at www.k12aleks.com.

Reading and Writing in Mathematics

Glencoe Algebra 2 provides numerous opportunities to incorporate reading and writing into the mathematics classroom.

Student Edition
- Foldables Study Organizer, p. 521
- Concept Check questions require students to verbalize and write about what they have learned in the lesson. (pp. 527, 535, 544, 549, 557, 563, 566)
- Writing in Math questions in every lesson, pp. 530, 537, 546, 551, 559, 564
- WebQuest, pp. 529, 565

Teacher Wraparound Edition
- Foldables Study Organizer, pp. 521, 566
- Study Notebook suggestions, pp. 522, 527, 535, 544, 549, 557, 563
- Modeling activities, pp. 530, 565
- Speaking activities, pp. 546, 559
- Writing activities, pp. 538, 551
- ELL Resources, pp. 520, 529, 537, 545, 550, 558, 564, 566

Additional Resources
- Vocabulary Builder worksheets require students to define and give examples for key vocabulary terms as they progress through the chapter. (Chapter 10 Resource Masters, pp. vii-viii)
- Reading to Learn Mathematics master for each lesson (Chapter 10 Resource Masters, pp. 577, 583, 589, 595, 601, 607)
- Vocabulary PuzzleMaker software creates crossword, jumble, and word search puzzles using vocabulary lists that you can customize.
- Teaching Mathematics with Foldables provides suggestions for promoting cognition and language.
- Reading and Writing in the Mathematics Classroom
- WebQuest and Project Resources

For more information on Reading and Writing in Mathematics, see pp. T6–T7.

For more information on Intervention and Assessment, see pp. T8–T11.
Have students read over the list of objectives and make a list of any words with which they are not familiar.

Point out to students that this is only one of many reasons why each objective is important. Others are provided in the introduction to each lesson.

**Key Vocabulary**
- exponential growth (p. 524)
- exponential decay (p. 524)
- logarithm (p. 531)
- common logarithm (p. 547)
- natural logarithm (p. 554)

**Exponential and Logarithmic Relations**

- **Lessons 10-1 through 10-3** Simplify exponential and logarithmic expressions.
- **Lessons 10-1, 10-4, and 10-5** Solve exponential equations and inequalities.
- **Lessons 10-2 and 10-3** Solve logarithmic equations and inequalities.
- **Lesson 10-6** Solve problems involving exponential growth and decay.

Exponential functions are often used to model problems involving growth and decay. Logarithms can also be used to solve such problems. You will learn how a declining farm population can be modeled by an exponential function in Lesson 10-1.

**Notes**

Vocabulary Builder

The Key Vocabulary list introduces students to some of the main vocabulary terms included in this chapter. For a more thorough vocabulary list with pronunciations of new words, give students the Vocabulary Builder worksheets found on pages vii and viii of the Chapter 10 Resource Masters. Encourage them to complete the definition of each term as they progress through the chapter. You may suggest that they add these sheets to their study notebooks for future reference when studying for the Chapter 10 test.
Prerequisite Skills To be successful in this chapter, you’ll need to master these skills and be able to apply them in problem-solving situations. Review these skills before beginning Chapter 10.

Lessons 10-1 through 10-3 Multiply and Divide Monomials Simplify. Assume that no variable equals 0. (For review, see Lesson 5-1.)
1. \(x^5 \cdot x \cdot x^6 x^{12}\)
2. \((3ab^2c^3)^2\)
3. \(-\frac{36c^2d^2}{21x^3y^4z^2}\)
4. \(-\frac{12a^3}{7f^2z}\)

Lessons 10-2 and 10-3 Solve Inequalities Solve each inequality. (For review, see Lesson 1-5)
5. \(a + 4 < -10\)
6. \(5n \leq 15\)
7. \(3y + 2 \geq -4\)
8. \(15 - x > 9\)

Lessons 10-2 and 10-3 Inverse Functions Find the inverse of each function. Then graph the function and its inverse. (For review, see Lesson 7-8.)
9. \(f^{-1}(x) = \frac{-1}{2}x\)
10. \(f(x) = 3x - 2\)
11. \(f(x) = -x + 1\)
12. \(f(x) = \frac{x - 4}{3}\)

Lessons 10-2 and 10-3 Composition of Functions Find \(g[h(x)]\) and \(h[g(x)]\). (For review, see Lesson 7-7.)
13. \(h(x) = 3x + 4\) \(g(h(x)) = 3x + 2\)
\(g(x) = x - 2\) \(h[g(x)] = 3x - 2\)
14. \(h(x) = 2x - 7\) \(g(h(x)) = 10x - 35\)
\(g(x) = 5x\) \(h[g(x)] = 10x - 7\)
15. \(h(x) = x - 4\) \(g(h(x)) = x^2 - 8x + 16\)
\(g(x) = x^2\) \(h[g(x)] = x^2 - 4\)
16. \(h(x) = 4x + 1\) \(g(h(x)) = -8x - 5\)
\(g(x) = -2x - 3\) \(h[g(x)] = -8x - 11\)

Fold and Cut

Fold in half along the width. On the first two sheets, cut along the fold at the ends. On the second two sheets, cut in the center of the fold as shown.

Fold and Label

Insert first sheets through second sheets and align folds. Label pages with lesson numbers.

Reading and Writing As you read and study the chapter, fill the journal with notes, diagrams, and examples for each lesson.

Organization of Data and Journal Writing After students make their Foldable journals, have them label two pages for each lesson in Chapter 10. Writers’ journals can be used by students to record the direction and progress of learning, to describe positive and negative experiences during learning, to write about personal associations and experiences called to mind during learning, and to list examples of ways in which new knowledge has or will be used in their daily life, as well as take notes, record key concepts, and write examples.
Investigating Exponential Functions

Collect the Data

Step 1 Cut a sheet of notebook paper in half.
Step 2 Stack the two halves, one on top of the other.
Step 3 Make a table like the one below and record the number of sheets of paper you have in the stack after one cut.

<table>
<thead>
<tr>
<th>Number of Cuts</th>
<th>Number of Sheets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Step 4 Cut the two stacked sheets in half, placing the resulting pieces in a single stack. Record the number of sheets of paper in the new stack after 2 cuts.
Step 5 Continue cutting the stack in half, each time putting the resulting piles in a single stack and recording the number of sheets in the stack. Stop when the resulting stack is too thick to cut.

Analyze the Data

1. Write a list of ordered pairs \((x, y)\), where \(x\) is the number of cuts and \(y\) is the number of sheets in the stack. Notice that the list starts with the ordered pair \((0, 1)\), which represents the single sheet of paper before any cuts were made.

2. Continue the list, beyond the point where you stopped cutting, until you reach the ordered pair for 7 cuts. Explain how you calculated the last \(y\) values for your list, after you had stopped cutting.
3. Plot the ordered pairs in your list on a coordinate grid. Be sure to choose a scale for the \(y\)-axis so that you can plot all of the points. See pp. 573A–573D.
4. Describe the pattern of the points you have plotted. Do they lie on a straight line? The points do not lie in a straight line. The slope increases as the \(x\) values increase.

Make a Conjecture

5. Write a function that expresses \(y\) as a function of \(x\). \(y = 2^x\)
6. Use a calculator to evaluate the function you wrote in Exercise 5 for \(x = 8\) and \(x = 9\). Does it give the correct number of sheets in the stack after 8 and 9 cuts?
7. Notebook paper usually stacks about 500 sheets to the inch. How thick would your stack of paper be if you had been able to make 9 cuts? About 1 in.
8. Suppose each cut takes about 5 seconds. If you had been able to keep cutting, you would have made 36 cuts in three minutes. At 500 sheets to the inch, make a conjecture as to how thick you think the stack would be after 36 cuts. Sample answer: 1 million ft
9. Use your function from Exercise 5 to calculate the thickness of your stack after 36 cuts. Write your answer in miles. 2169 mi

Have students work in small groups for Exercises 1–9. Observe students’ work to determine if they are able to write the function in Exercise 5. Students should conclude after Exercise 9 that exponential functions can increase faster than seems reasonable.
**What You’ll Learn**

- Graph exponential functions.
- Solve exponential equations and inequalities.

**Vocabulary**

- exponential function
- exponential growth
- exponential decay
- exponential equation
- exponential inequality

**Study Tip**

Common Misconception

Be sure not to confuse polynomial functions and exponential functions. While \( y = x^2 \) and \( y = 2^x \) each have an exponent, \( y = x^2 \) is a polynomial function and \( y = 2^x \) is an exponential function.

**Example 1 Graph an Exponential Function**

Sketch the graph of \( y = 2^x \). Then state the function’s domain and range.

Make a table of values. Connect the points to sketch a smooth curve.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>( 2^{-3} = \frac{1}{8} )</td>
</tr>
<tr>
<td>-2</td>
<td>( 2^{-2} = \frac{1}{4} )</td>
</tr>
<tr>
<td>-1</td>
<td>( 2^{-1} = \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>( 2^0 = 1 )</td>
</tr>
<tr>
<td>( \frac{1}{2} )</td>
<td>( 2^{\frac{1}{2}} = \sqrt{2} )</td>
</tr>
<tr>
<td>1</td>
<td>( 2^1 = 2 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2^2 = 4 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2^3 = 8 )</td>
</tr>
</tbody>
</table>

The domain is all real numbers, while the range is all positive numbers.

**EXPONENTIAL FUNCTIONS**

In an exponential function like \( y = 2^x \), the base is a constant, and the exponent is a variable. Let’s examine the graph of \( y = 2^x \).

**Example 1**

Graph the exponential function \( y = 2^x \). Then state the function’s domain and range.

Make a table of values. Connect the points to sketch a smooth curve.

The domain is all real numbers, while the range is all positive numbers.

**5-Minute Check Transparency 10-1** Use as a quiz or review of Chapter 9.

**Mathematical Background** notes are available for this lesson on p. 520C.

**Building on Prior Knowledge**

Ask students where they have heard the term *exponential* before and what they think it might mean. Students may have heard terms like exponential growth on a television news program and they might think that exponential means “enormous.” Use students’ answers to introduce the concept of exponential functions.

**How does an exponential function describe tournament play?**

Ask students:

- How many winners are there in the first round of the tournament? 32
- After each round, how has the number of teams changed? The number of teams remaining after each round is half the number of teams that played in that round.
- If the tournament field was reduced to 32 teams, how many basketball games would have to be played by the tournament’s winning team? 5 games
Answers

1. The shapes of the graphs are the same.
2. The asymptote for each graph is the x-axis and the y-intercept for each graph is 1.
3. The graphs are reflections of each other over the y-axis.
4. The graphs are reflections of each other over the x-axis.

In general, an equation of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$, is called an exponential function with base $b$. Exponential functions have the following characteristics:

1. The function is continuous and one-to-one.
2. The domain is the set of all real numbers.
3. The x-axis is an asymptote of the graph.
4. The range is the set of all positive numbers if $a > 0$ and all negative numbers if $a < 0$.
5. The graph contains the point $(0, a)$. That is, the y-intercept is $a$.
6. The graphs of $y = ab^x$ and $y = a \left( \frac{1}{b} \right)^x$ are reflections across the y-axis.

There are two types of exponential functions: exponential growth and exponential decay.

The base of an exponential growth function is a number greater than one. The base of an exponential decay function is a number between 0 and 1.

- If $a > 0$ and $b > 1$, the function $y = ab^x$ represents exponential growth.
- If $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents exponential decay.

You can use a TI-83 Plus graphing calculator to look at the graph of two other exponential functions, $y = 3x$ and $y = \left( \frac{1}{3} \right)^x$. 

### Graphing Calculator Investigation

#### Families of Exponential Functions

Have students begin by graphing the two functions separately, so they recognize that the two curves shown in the book are two distinct graphs. Students are used to seeing the U-shaped graphs of polynomial functions and might have difficulty separating the graphs visually. Also, a reminder about the meaning of the term asymptotes may be helpful for many students.
Example 2 Identify Exponential Growth and Decay

Determine whether each function represents exponential growth or decay.

<table>
<thead>
<tr>
<th>Function</th>
<th>Exponential Growth or Decay?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. ( y = \left( \frac{1}{5} \right)^x )</td>
<td>The function represents exponential decay, since the base, ( \frac{1}{5} ), is between 0 and 1.</td>
</tr>
<tr>
<td>b. ( y = 3(4)^x )</td>
<td>The function represents exponential growth, since the base, 4, is greater than 1.</td>
</tr>
<tr>
<td>c. ( y = 7(1.2)^x )</td>
<td>The function represents exponential growth, since the base, 1.2, is greater than 1.</td>
</tr>
</tbody>
</table>

Exponential functions are frequently used to model the growth or decay of a population. You can use the \( y \)-intercept and one other point on the graph to write the equation of an exponential function.

Example 3 Write an Exponential Function

Farming

In 1983, there were 102,000 farms in Minnesota, but by 1998, this number had dropped to 80,000.

a. Write an exponential function of the form \( y = ab^x \) that could be used to model the farm population \( y \) of Minnesota. Write the function in terms of \( x \), the number of years since 1983.

For 1983, the time \( x \) equals 0, and the initial population \( y \) is 102,000. Thus, the \( y \)-intercept, and value of \( a \), is 102,000.

For 1998, the time \( x \) equals 1998 – 1983 or 15, and the population \( y \) is 80,000. Substitute these values and the value of \( a \) into an exponential function to approximate the value of \( b \).

\[
y = ab^x \quad \text{Exponential function}
\]

\[
80,000 = 102,000b^{15} \quad \text{Replace } x \text{ with 15, } y \text{ with 80,000, and } a \text{ with 102,000.}
\]

\[
0.78 = b^{15} \quad \text{Divide each side by 102,000.}
\]

\[
\sqrt[15]{0.78} = b \quad \text{Take the 15th root of each side.}
\]

To find the 15th root of 0.78, use selection 5: \( \sqrt{} \) under the MATH menu on the TI-83 Plus.

**KEYSTROKES:** 15 [MATH] 5 0.78 ENTER 9835723396

An equation that models the farm population of Minnesota from 1983 to 1998 is

\[
y = 102,000(0.98)^x
\]

b. Suppose the number of farms in Minnesota continues to decline at the same rate. Estimate the number of farms in 2010.

For 2010, the time \( x \) equals 2010 – 1983 or 27.

\[
y = 102,000(0.98)^{27} \quad \text{Modeling equation}
\]

\[
y = 102,000(0.98)^{27} \quad \text{Replace } x \text{ with 27.}
\]

\[
y \approx 59,115 \quad \text{Use a calculator.}
\]

The farm population in Minnesota will be about 59,115 in 2010.
In-Class Examples

4. Simplify each expression.
   a. \(5\sqrt[3]{2} + 5\sqrt[3]{2} - \sqrt[3]{2}\)
   b. \((6\sqrt[3]{2})\sqrt[6]{6}\)

5. Solve each equation.
   a. \(4^{9n} - 2 = 256\) \(n = \frac{2}{3}\)
   b. \(3^{5x} = 9^{2x - 1}\) \(x = -2\)

Study Tip

Look Back
To review Properties of Power, see Lesson 5-1.

Example 4 Simplify Expressions with Irrational Exponents

Simplify each expression.

a. \(2\sqrt[5]{3} \cdot 2\sqrt[5]{3} = 2\sqrt[5]{3^2} = 2\sqrt[5]{9}\)
   Property of Powers

b. \((7\sqrt[2]{3})\sqrt[3]{3}\) \(7\sqrt[2]{3^3} = 7\sqrt[2]{27}\)
   Power of a Power

The following property is useful for solving exponential equations. Exponential equations are equations in which variables occur as exponents.

Key Concept Property of Equality for Exponential Functions

- **Symbols** If \(b\) is a positive number other than 1, then \(b^x = b^y\) if and only if \(x = y\).
- **Example** If \(2^x = 2^8\), then \(x = 8\).

Example 5 Solve Exponential Equations

Solve each equation.

a. \(3^{2n} + 1 = 81\)
   
   \[3^{2n} + 1 = 81\] Original equation
   \[3^{2n} + 1 = 3^4\] Rewrite 81 as \(3^4\) so each side has the same base.
   \[2n + 1 = 4\] Property of Equality for Exponential Functions
   \[2n = 3\] Subtract 1 from each side.
   \[n = \frac{3}{2}\] Divide each side by 2.

   The solution is \(\frac{3}{2}\).

   **CHECK** \(3^{2n} + 1 = 81\) Original equation
   \[3^{\left(\frac{3}{2}\right)} + 1 = 81\] Substitute \(\frac{3}{2}\) for \(n\).
   \[3^3 = 81\] Simplify.
   \[81 = 81\] Simplify.

b. \(4^{2x} = 8^{x - 1}\)
   
   \[4^{2x} = 8^{x - 1}\] Original equation
   \[(2^2)^{2x} = (2^3)^{x - 1}\] Rewrite each side with a base of 2.
   \[2^{4x} = 2^{3x - 3}\] Power of a Power
   \[4x = 3(x - 1)\] Property of Equality for Exponential Functions
   \[4x = 3x - 3\] Distributive Property
   \[x = -3\] Subtract 3x from each side.

   The solution is \(-3\).
The following property is useful for solving inequalities involving exponential functions or exponential inequalities.

**Key Concept**  **Property of Inequality for Exponential Functions**
- **Symbols**  If $b > 1$, then $b^x > b^y$ if and only if $x > y$, and $b^y < b^x$ if and only if $x < y$.
- **Example**  If $5^x < 5^y$, then $x < 4$.

This property also holds for $\leq$ and $\geq$.

### Example 6  Solve Exponential Inequalities

Solve $4^{3p} - 1 > \frac{1}{256}$.

1. Original inequality
2. $4^{3p} - 1 > \frac{1}{256}$
3. Rewrite $\frac{1}{256}$ as $\frac{1}{4^4}$ or $4^{-4}$ so each side has the same base.
4. $3p - 1 > -4$
5. Apply the Property of Inequality for Exponential Functions
6. $3p > -3$
7. Add 1 to each side.
8. $p > -1$
9. Divide each side by 3.

The solution set is $p > -1$.

**CHECK**

Test a value of $p$ greater than $-1$; for example, $p = 0$.

1. $4^{3p} - 1 > \frac{1}{256}$  Original inequality
2. $4^{3(0)} - 1 > \frac{1}{256}$  Replace $p$ with 0.
3. $4^{-1} > \frac{1}{256}$  Simplify.
4. $\frac{1}{4} > \frac{1}{256}$  $a^{-1} = \frac{1}{a}$

### Check for Understanding

**Concept Check**

1. **OPEN ENDED**  Give an example of a value of $b$ for which $y = b^x$ represents exponential decay.  **Sample answer:** 0.8

2. **Identify** each function as linear, quadratic, or exponential.
   - a. $y = 3x^2$
   - b. $y = 4(3)^x$
   - c. $y = 2x + 4$
   - d. $y = 4(0.2)^x + 1$

3. **Match** each function with its graph.
   - a. $y = 5^x$
   - b. $y = 2(5)^x$
   - c. $y = (\frac{1}{5})^x$

   **Guided Practice**

6-7. See margin.

### Answers

6. $D = \{x \mid x$ is all real numbers.$\}$, $R = \{y \mid y > 0\}$

7. $D = \{x \mid x$ is all real numbers.$\}$, $R = \{y \mid y > 0\}$

---

**Auditory/Musical**  Going around the room, have students count by ones beginning at 2, with each student calling out one number. Instruct them to record the number they called as $n$. Then have students find $n^2$ and $2^n$. Now go around the room again and ask students to state their value of $n^2$ (for a class of 30 students, the recited numbers are all the squares from 4 to 961). Now have students state their values of $2^n$ (for a class of 30, the recited numbers are all the powers of 2 from 4 to $2^{31}$ or about $2 \times 10^9$).
Determine whether each function represents exponential growth or decay.

8. \( y = 2(7)^x \) growth 9. \( y = (0.5)^x \) decay 10. \( y = 0.3(5)^x \) growth

Write an exponential function whose graph passes through the given points.

11. \((0, 3)\) and \((-1, 6)\) \( y = 3 \left( \frac{1}{2} \right)^x \) 12. \((0, -18)\) and \((-2, -2)\) \( y = -18(3)^x \)

Simplify each expression.

13. \(2\sqrt{2} \cdot 2\sqrt{7} \) or \(4\sqrt{7} \) 14. \((a^n)^4 \) \(a^{4n} \) 15. \(81\sqrt{2} \div 3\sqrt{2} \) \(3\sqrt{2} \) or \(27\sqrt{2} \)

Solve each equation or inequality. Check your solution.

16. \(2^x + 4 = \frac{1}{2} \) \(x = 0\) 17. \(5^{2x} + 3 \leq 125 \) \(x \leq 0\) 18. \(9^{2y - 1} = 27y \) \(2y \)

ANIMAL CONTROL For Exercises 19 and 20, use the following information.

During the 19th century, rabbits were brought to Australia. Since the rabbits had no natural enemies on that continent, their population increased rapidly. Suppose there were 65,000 rabbits in Australia in 1865 and 2,500,000 in 1867.

19. Write an exponential function that could be used to model the rabbit population \(y\) in Australia. Write the function in terms of \(x\), the number of years since 1865.

\[ y = 65,000(6.20)^x \]

20. Assume that the rabbit population continued to grow at that rate. Estimate the Australian rabbit population in 1872.

\[ 22,890,495,000 \]

Answers (p. 529)

60. 9.67 million; 17.62 million; 32.12 million; These answers are in close agreement with the actual populations in those years.

61. 2144.97 million; 281.42 million; No, the growth rate has slowed considerably. The population in 2000 was much smaller than the equation predicts it would be.
For Exercises 57 and 58, use the following information.
The number of bacteria in a colony is growing exponentially.
57. Write an exponential function to model the population y of bacteria x hours after 2 P.M. \( y = 100(6.32)^x \)
58. How many bacteria were there at 7 P.M. that day? about 1,008,290

For Exercises 59–61, use the following information.
Every ten years, the Bureau of the Census counts the number of people living in the United States. In 1790, the population of the U.S. was 3.93 million. By 1800, this number had grown to 5.31 million.
59. Write an exponential function that could be used to model the U.S. population \( y \) in millions for 1790 to 1800. Write the equation in terms of \( x \), the number of decades \( x \) since 1790. \( y = 3.93(1.35)^x \)
60. Assume that the U.S. population continued to grow at that rate. Estimate the population for the years 1820, 1840, and 1860. Then compare your estimates with the actual population for those years, which were 9.64, 17.06, and 31.44 million, respectively. See margin.
61. RESEARCH Estimate the population of the U.S. in 2000. Then use the Internet or other reference to find the actual population of the U.S. in 2000. Has the population of the U.S. continued to grow at the same rate at which it was growing in the early 1800s? Explain. See margin.

For Exercises 62–64, use the following information.
Suppose you deposit a principal amount of \( P \) dollars in a bank account that pays compound interest. If the annual interest rate is \( r \) (expressed as a decimal) and the bank makes interest payments \( n \) times every year, the amount of money \( A \) you would have after \( t \) years is given by \( A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \).
62. If the principal, interest rate, and number of interest payments are known, what type of function is \( A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \)? Explain your reasoning.
63. Write an equation giving the amount of money you would have after \( t \) years if you deposit $1000 into an account paying 4% annual interest compounded quarterly (four times per year). \( A(t) = 1000(1.01)^{4t} \)
64. Find the account balance after 20 years. $2216.72

For Exercises 65 and 66, use the information at the left.
65. If a typical computer operates with a computational speed \( s \) today, write an expression for the speed at which you can expect an equivalent computer to operate after \( x \) three-year periods. \( s \cdot 4^x \)
66. Suppose your computer operates with a processor speed of 600 megahertz and you want a computer that can operate at 4800 megahertz. If a computer with that speed is currently unavailable for home use, how long can you expect to wait until you can buy such a computer? 1.5 three-year periods or 4.5 yr

For Exercises 67 and 68, use the following information.
7. CRITICAL THINKING Decide whether the following statement is sometimes, always, or never true. Explain your reasoning.
For a positive base \( b \) other than 1, \( b^x > b^y \) if and only if \( x > y \).

Enrichment, p. 578
Finding Solutions of \( x^a = y^b \)
Perhaps you have noticed that if \( a \) and \( b \) are integers and \( x = y \), the resulting equation is equivalent to the original equation. This means that the solutions of \( x^a = y^b \) and drawing its graph is not a simple process.
Solve each problem. Assume that \( a \) and \( b \) are positive real numbers.
1. If \( x > 0 \) will it be a solution of \( x^a = y^b \)? Justify your answer.
Yes, since \( a^x = b^y \) must be true (Refl. Prop. of Equality).
2. If \( x = 0 \), \( x^a = y^0 \), and if \( y = 0 \), \( x^0 = y^b \). Is \( x = 0 \) also a solution? Justify your answer.
Yes; replacing a with d, b with c gives \( d^e = c^f \), but if \( c \), \( d \) in a solution, \( c^f = d^e \). So, by the Symmetric Property of Equality, \( d^e = c^f \) is true.
68. **Writing in Math**  Answer the question that was posed at the beginning of the lesson.  See pp. 573A–573D.

How does an exponential function describe tournament play? Include the following in your answer:
- an explanation of how you could use the equation $y = 2^x$ to determine the number of rounds of tournament play for 128 teams, and
- an example of an inappropriate number of teams for tournament play with an explanation as to why this number would be inappropriate.

69. If $4^x + 2 = 48$, then $4^x = A$
   - $A$  3.0.  
   - B  6.4.  
   - C  6.9.  
   - D  12.0.  
   - E  24.0.

70. **GRID IN**  Suppose you deposit $500 in an account paying 4.5% interest compounded semiannually. Find the dollar value of the account rounded to the nearest penny after 10 years.  **780.25**

### PREREQUISITE SKILL  Getting Ready for Lesson 10-2

**Getting Ready for Lesson 10-2**

**PREREQUISITE SKILL**  In Lesson 10-2, students will evaluate logarithmic expressions. Because logarithmic and exponential functions are inverses of each other, their composites are the identity function. Students must be familiar with compositions of functions in order to evaluate these inverse functions. Use Exercises 87–89 to determine your students’ familiarity with composition of functions.

### Answers

75. For $h > 0$, the graph of $y = 2^x$ is translated $|h|$ units to the right. For $h < 0$, the graph of $y = 2^x$ is translated $|h|$ units to the left. For $k > 0$, the graph of $y = 2^x$ is translated $|k|$ units up. For $k < 0$, the graph of $y = 2^x$ is translated $|k|$ units down.

**80–82. See margin for graphs.**

80. $y = \sqrt{x-2}$

81. $y = -2|x|

82. $y = 2x$  

**86. ENERGY**  A circular cell must deliver 18 watts of energy. If each square centimeter of the cell that is in sunlight produces 0.01 watt of energy, how long must the radius of the cell be?  **(Lesson 5-8)**  about 23.94 cm

**87.**  $h(x) = 2x - 1$  
   - $g(x) = x - 5$

88. $h(x) = x + 3$  
   - $g(x) = x^2$

89. $h(x) = 2x + 5$  
   - $g(x) = -x + 3$

87. $g[h(x)] = 2x - 6$; $h[g(x)] = 2x - 11$

88. $g[h(x)] = x^2 + 6x + 9$; $h[g(x)] = x^2 + 3$

89. $g[h(x)] = -2x - 2$; $h[g(x)] = -2x + 11$

88–89. See margin.
Logarithms and Logarithmic Functions

What You'll Learn
• Evaluate logarithmic expressions.
• Solve logarithmic equations and inequalities.

Why is a logarithmic scale used to measure sound?
Many scientific measurements have such an enormous range of possible values that it makes sense to write them as powers of 10 and simply keep track of their exponents. For example, the loudness of sound is measured in units called decibels. The graph shows the relative intensities and decibel measures of common sounds.

LOGARITHMIC FUNCTIONS AND EXPRESSIONS To better understand what is meant by a logarithm, let's look at the graph of \( y = 2^x \) and its inverse. Since exponential functions are one-to-one, the inverse of \( y = 2^x \) exists and is also a function. Recall that you can graph the inverse of a function by interchanging the \( x \) and \( y \) values in the ordered pairs of the function.

The inverse of \( y = 2^x \) can be defined as \( x = 2^y \). Notice that the graphs of these two functions are reflections of each other over the line \( y = x \).

In general, the inverse of \( y = b^x \) is \( x = b^y \). In \( x = b^y \), \( y \) is called the logarithm of \( x \). It is usually written as \( y = \log_b x \) and is read \( y \) equals \( \log \) base \( b \) of \( x \).

Vocabulary
• logarithm
• logarithmic function
• logarithmic equation
• logarithmic inequality

Study Tip
Look Back
To review inverse functions, see Lesson 7-8.

5-Minute Check Transparency 10-2 Use as a quiz or review of Lesson 10-1.
Mathematical Background notes are available for this lesson on p. 520C.

Ask students:
• On the number line shown, the scale along the bottom is 10 decibels per tick mark. What do you notice about the scale along the top for relative intensity?
  The scale is not uniform; the relative intensity at the first tick mark is 10, at the second it is 100, at the third it is 1000, and so on.
• If you draw a number line with a uniform scale whose tick marks are labeled from 0 to \( 10^{12} \), what number is at the midpoint between 0 to \( 10^{12} \)? \( \times 10^{11} \)
• Where does the point 1 million appear on your number line? very close to the point for 0
• Where does the point 100 appear on your number line? very close to the point for 0
• What problem arises with trying to represent the relative intensities on a standard number line? Sample answer: The lesser intensities are so close together near 0 on the number line that they are difficult to represent accurately.
LOGARITHMIC FUNCTIONS AND EXPRESSIONS

Teaching Tip After discussing the definition of logarithm at the bottom of p. 531, write the equation \( y = 2x \) on the chalkboard and ask students to rewrite the equation with \( x \) in terms of \( y \). Repeat this for the equation \( y = x^2 \). Now write the equation \( y = 2^x \) on the chalkboard and ask students to rewrite this equation with \( x \) in terms of \( y \). This will likely have students stymied. Explain that the rewritten equation is \( x = \log_2 y \).

In-Class Examples

1. Write each equation in exponential form.
   a. \( \log_3 9 = 2 \) so \( 3^2 = 9 \)
   b. \( \log_{10} \frac{1}{100} = -2 \) so \( 10^{-2} = \frac{1}{100} \)

2. Write each equation in logarithmic form.
   a. \( 5^3 = 125 \) so \( \log_5 125 = 3 \)
   b. \( 2^{\frac{1}{3}} = 3 \) so \( \log_2 3 = \frac{1}{3} \)

3. Evaluate \( \log_3 243 \). \( 5 \)

Study Tip

Look Back
To review composition of functions, see Lesson 7-7.

Example 1 Logarithmic to Exponential Form
Write each equation in exponential form.

a. \( \log_5 1 = 0 \)
   \( \log_5 1 = 0 \rightarrow 1 = 5^0 \)

b. \( \log_2 \frac{1}{16} = -4 \)
   \( \log_2 \frac{1}{16} = -4 \rightarrow \frac{1}{16} = 2^{-4} \)

Example 2 Exponential to Logarithmic Form
Write each equation in logarithmic form.

a. \( 10^3 = 1000 \)
   \( 10^3 = 1000 \rightarrow \log_{10} 1000 = 3 \)

b. \( 9^{\frac{1}{2}} = 3 \)
   \( 9^{\frac{1}{2}} = 3 \rightarrow \log_9 3 = \frac{1}{2} \)

Example 3 Evaluate Logarithmic Expressions
Evaluate \( \log_2 64 \).

\( \log_2 64 = y \) Let the logarithm equal \( y \).

\( 64 = 2^y \) Definition of logarithm

\( 2^y = 2^x \)

\( 64 = 2^x \)

\( 6 = y \) Property of Equality for Exponential Functions

So, \( \log_2 64 = 6 \).

The function \( y = \log_b x \), where \( b > 0 \) and \( b \neq 1 \), is called a logarithmic function.

As shown in the graph on the previous page, this function is the inverse of the exponential function \( y = b^x \) and has the following characteristics.
1. The function is continuous and one-to-one.
2. The domain is the set of all positive real numbers.
3. The \( y \)-axis is an asymptote of the graph.
4. The range is the set of all real numbers.
5. The graph contains the point \((1, 0)\). That is, the \( x \)-intercept is 1.

Since the exponential function \( f(x) = b^x \) and the logarithmic function \( g(x) = \log_b x \) are inverses of each other, their composites are the identity function. That is, \( f(g(x)) = x \) and \( g(f(x)) = x \).

\[
\begin{align*}
f(g(x)) &= x & g(f(x)) &= x \\
f(log_b x) &= x & g(b^x) &= x \\
f(log_b x) &= x & \log_b b^x &= x
\end{align*}
\]
Thus, if their bases are the same, exponential and logarithmic functions “undo” each other. You can use this inverse property of exponents and logarithms to simplify expressions.

**Example 4**  Inverse Property of Exponents and Logarithms

Evaluate each expression.

a. $\log_6 6^8$
   
   $\log_6 6^8 = 8 \quad \text{log base b of x = } x$

b. $3\log_3 (4x - 1)$
   
   $3\log_3 (4x - 1) = 4x - 1 \quad 3^{\log_3 x} = x$

**SOLVE LOGARITHMIC EQUATIONS AND INEQUALITIES**

A **logarithmic equation** is an equation that contains one or more logarithms. You can use the definition of a logarithm to help you solve logarithmic equations.

**Example 5**  Solve a Logarithmic Equation

Solve $\log_4 n = \frac{5}{2}$

$\log_4 n = \frac{5}{2}$  
Original equation

$n = 4^{\frac{5}{2}}$  
Definition of logarithm

$n = (2^2)^{\frac{5}{2}} = 2^5$  
Power of a Power

$n = 32$  
Simplify.

A **logarithmic inequality** is an inequality that involves logarithms. In the case of inequalities, the following property is helpful.

**Key Concept**  Logarithmic to Exponential Inequality

- **Symbols**
  - If $b > 1$, $x > 0$, and $\log_b x > y$, then $x > b^y$.
  - If $b > 1$, $x > 0$, and $\log_b x < y$, then $0 < x < b^y$.

- **Examples**
  - $\log_2 x > 3$
    
    $x > 2^3$
  - $\log_3 x < 5$
    
    $0 < x < 3^5$

**Example 6**  Solve a Logarithmic Inequality

Solve $\log_5 x < 2$. Check your solution.

$\log_5 x < 2$  
Original inequality

$0 < x < 5^2$  
Logarithmic to exponential inequality

$0 < x < 25$  
Simplify.

The solution set is $\{x | 0 < x < 25\}$.

**CHECK**

Try 5 to see if it satisfies the inequality.

$\log_5 5 < 2$  
Substitute 5 for $x$.

$1 < 2 \checkmark \quad \log_5 5 = 1$ because $5^1 = 5$.  

www.algebra2.com/extra_examples
Exponential and Logarithmic Relations

In-Class Examples

7 Solve $\log_{4} x^2 = \log_{4} (4x - 3)$. Check your solution. $1, 3$

8 Solve $\log_{7} (2x + 8) > \log_{7} (x + 5)$. Check your solution. $x > -3$

Intervention
Students have not covered logarithmic functions before and are likely to find them confusing. Expect students to need extra time to absorb the material in this lesson before continuing with the rest of the chapter.

Example 7 Solve Equations with Logarithms on Each Side

Solve $\log_{5} (p^2 - 2) = \log_{5} p$. Check your solution.

Original equation
$p^2 - 2 = p$
Subtract $p$ from each side.
$p^2 - p - 2 = 0$
Factor.
$(p - 2)(p + 1) = 0$
$p - 2 = 0$ or $p + 1 = 0$
Zero Product Property
$p = 2$ or $p = -1$
Solve each equation.

CHECK Substitute each value into the original equation.

$log_{5} (2^2 - 2) = log_{5} 2$
Substitute 2 for $p$.
$log_{5} 2 = log_{5} 2$
Simplify.

$log_{5} \left[ (-1)^2 - 2 \right] = log_{5} (-1)$
Substitute $-1$ for $p$.
Since $log_{5} (-1)$ is undefined, $-1$ is an extraneous solution and must be eliminated. Thus, the solution is 2.

Key Concept Property of Equality for Logarithmic Functions

- **Symbols**
  - If $b$ is a positive number other than 1, then $\log_{b} x = \log_{b} y$ if and only if $x = y$.

- **Example**
  - If $\log_{7} x = \log_{7} 3$, then $x = 3$.

Example 8 Solve Inequalities with Logarithms on Each Side

Solve $\log_{10} (3x - 4) < \log_{10} (x + 6)$. Check your solution.

Original inequality
$3x - 4 < x + 6$
Addition and Subtraction Properties of Inequalities
$2x < 10$
Divide each side by 2.

We must exclude from this solution all values of $x$ such that $3x - 4 \leq 0$ or $x + 6 \leq 0$. Thus, the solution set is $x > \frac{4}{3}$ and $x > -6$ and $x < 5$. This compound inequality simplifies to $\frac{4}{3} < x < 5$.

Key Concept Property of Inequality for Logarithmic Functions

- **Symbols**
  - If $b > 1$, then $\log_{b} x > \log_{b} y$ if and only if $x > y$, and $\log_{b} x < \log_{b} y$ if and only if $x < y$.

- **Example**
  - If $\log_{2} x > \log_{2} 9$, then $x > 9$.

This property also holds for $\leq$ and $\geq$.

D A I L Y

Intervention

Visual/Spatial Have students create colorful posters showing several equivalent exponential and logarithmic equations, such as $2^3 = 8$ and $3 = \log_{2} 8$. Suggest that students use a different color for each of the digits 2, 3, and 8 to help them visualize the relative locations of the digits in the pairs of equations.
Check for Understanding

**Guided Practice**

1. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation. Sample answer: $y = 5^x$ and $x = \log_5 y$

2. **Describe** the relationship between $y = 3^x$ and $y = \log_3 x$. They are inverses.

3. **FIND THE ERROR** Paul and Scott are solving $\log_3 x = 9$.

   **Paul**
   
   $\log_3 x = 9$
   
   $3^9 = x$
   
   $x = 19,683$

   **Scott**
   
   $\log_3 x = 9$
   
   $x = 3^9$
   
   $x = 19,683$

Who is correct? Explain your reasoning. **Scott**; see margin for explanation.

**Application**

**SOUND** For Exercises 18–20, use the following information.

An equation for loudness $L$, in decibels, is $L = 10 \log_{10} R$, where $R$ is the relative intensity of the sound.

18. Solve $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels. $10^{13}$

19. Solve $75 = 10 \log_{10} R$ to find the relative intensity of a concert with a loudness of 75 decibels. $10^{7.5}$

20. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities. $10^{5.5}$ or about 316,228 times

**Concept Check**

1. **OPEN ENDED** Give an example of an exponential equation and its related logarithmic equation. Sample answer: $y = 5^x$ and $x = \log_5 y$

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   **Scott**
   
   $\log_3 x = 9$
   
   $x = 3^9$
   
   $x = 19,683$

Who is correct? Explain your reasoning. **Scott**; see margin for explanation.

**Guided Practice**

4. $5^4 = 625$ $\log_5 625 = 4$

5. $7^{-2} = \frac{1}{49}$ $\log_7 \frac{1}{49} = -2$

**Write each equation in logarithmic form.**

6. $\log_3 81 = 4$ $3^4 = 81$

7. $\log_{56} 6 = \frac{3}{2}$ $36^2 = 6$

**Evaluate each expression.**

8. $\log_5 256$ $4$

9. $\log_2 \frac{1}{8}$ $-3$

10. $3^{\log_3 21}$ $21$

11. $\log_5 5^{-1}$ $-1$

12. $\log_9 x = \frac{3}{2}$ $27$

13. $\log_3 x = -3$ $1000$

14. $\log_3 (2x - 1) = 2$ $1 < x \leq 5$

15. $\log_5 (3x - 1) = \log_5 2x^2$ $\frac{1}{2}$

16. $\log_2 (3x - 5) > \log_2 (x + 7)$ $x > 6$

17. $\log_8 9 = 2$ $3$

**Solve each equation or inequality.** Check your solutions.

18. $130 = 10 \log_{10} R$ to find the relative intensity of a fireworks display with a loudness of 130 decibels. $10^{13}$

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**Guided Practice**

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20. How many times more intense is the fireworks display than the concert? In other words, find the ratio of their intensities. $10^{5.5}$ or about 316,228 times

**Answer**

3. The value of a logarithmic equation, 9, is the exponent of the equivalent exponential equation, and the base of the logarithmic expression, 3, is the base of the exponential equation. Thus $x = 3^9$, which cannot be true.

**Online Lesson Plans**

USA TODAY Education’s Online site offers resources and interactive features connected to each day’s newspaper. Experience TODAY, USA TODAY’s daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.
Write each equation in logarithmic form. 23. $\log_5 125 = -3$
24. $\left(\frac{1}{2}\right)^{-2} = 9$
25. $\log_3 9 = -2$
26. $\log_{10} 100^3 = 10$
27. $\log_5 125 = 3$
28. $\log_{10} 169 = 2$
29. $\log_4 1 = -1$
30. $\log_{100} 100^2 = -1$
31. $\log_8 4 = \frac{2}{3}$
32. $\log_5 25 = -2$
33. $\log_2 16 = 4$
34. $\log_{12} 144 = 2$
35. $\log_{16} 4 = \frac{1}{2}$
36. $\log_9 243 = \frac{5}{2}$
37. $\log_2 \frac{1}{32} = -5$
38. $\log_3 \frac{1}{81} = -4$
39. $\log_5 5^7 = 7$
40. $2 \log_5 45$
41. $\log_{11} 11^{n-5} = n - 5$
42. $6 \log_6 (3x + 2) = 3x + 2$
43. $\log_{10} 0.001 = -3$
44. $\log_4 16^2 = 2x$

WORLD RECORDS For Exercises 45 and 46, use the information given for Exercises 18–20 to find the relative intensity of each sound. Source: The Guinness Book of Records

45. The loudest animal sounds are the low-frequency pulses made by blue whales when they communicate. These pulses have been measured up to 188 decibels. 10^{18.8}
46. The loudest insect is the African cicada. It produces a calling song that measures 106.7 decibels at a distance of 50 centimeters. 10^{6.7}

Solve each equation or inequality. Check your solutions.
47. $\log_9 x = 2$
48. $\log_2 c > 8$
49. $\log_{64} y \leq \frac{1}{2}$
50. $\log_{25} n = \frac{3}{2}$
51. $\log_7 x = -1$
52. $\log_3 p < 0$
53. $\log_2 (3x - 8) \geq 6$
54. $\log_{10} (x^2 + 1) = 1$
55. $\log_6 64 = 3$
56. $\log_6 121 = 2$
57. $\log_5 5^{3x+1} = 13$
58. $\log_5 x = \frac{1}{2} \sqrt{5}$
59. $\log_6 (2x - 3) = \log_6 (x + 2)$
60. $\log_2 (4y - 10) \geq \log_2 (y - 1)$
61. $\log_5 (a^2 - 6) > \log_5 a + 3$
62. $\log_7 (x^2 + 36) = \log_7 100$

Show that each statement is true. 63–65. See margin.
63. $\log_5 25 = 2 \log_5 5$
64. $\log_{16} 2 \cdot \log_2 16 = 1$
65. $\log_5 [\log_3 (\log_2 8)] = 0$

Answers

63. $\log_5 25 \neq 2 \log_5 5$
64. $\log_{16} 2 \cdot \log_2 16 \neq 1$
65. $\log_7 [\log_3 (\log_2 8)] \neq 0$

Original equation $8 = 2^3$
Inverse Property of Exponents and Logarithms $3 = 3^1$
Inverse Property of Exponents and Logarithms $1 = 7^0$
Inverse Property of Exponents and Logarithms $0 = 0$

66. Sketch the graphs of \( y = \log_2 x \) and \( y = \left(\frac{1}{2}\right)^x \) on the same axes.

67. Sketch the graphs of \( y = \log_2 x + 3, \ y = \log_2 x - 4, \ y = \log_2 (x - 1), \) and \( y = \log_2 (x + 2). \)

68. Sketch the family of graphs in terms of its parent graph \( y = \log_2 x. \)

---

**EARTHQUAKE**

69. How many times as great is the amplitude caused by an earthquake with a magnitude of 7.2 compared to one with a magnitude of 6.9? The function is written as \( \frac{10^3}{10^2} \) or 100 as times greater than 10.

70. **NOISE ORDINANCE**

A proposed city ordinance will make it illegal to create sound in a residential area that exceeds 72 decibels during the day and 55 decibels during the night. How many times more intense is the noise level allowed during the day than at night? \(10^{1.7}\) or about 25 times as great.

---

**CRITICAL THINKING**

71. The value of \( \log_2 8 \) is between two consecutive integers. Name these integers and explain how you determined them.

72. Using the definition of a logarithmic function where \( y = \log_b x, \) explain why the base \( b \) cannot equal 1. All powers of 1 are 1, so the inverse of \( y = x^1 \) is not a function.

73. **WRITING IN MATH**

Answer the question that was posed at the beginning of the lesson. See pp. 573A–573D.

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**Skills Practice, p. 581 and Practice, p. 582 (shown)**

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**Reading to Learn Mathematics, p. 583**

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**Enrichment, p. 584**

**Musical Relationships**

The frequency of notes on a musical scale that are one octave apart are related by a numerical ratio. The general equation \( f_2 = \frac{f_1}{2} \), where the subscript 1 represents the frequency of note 1.

1. Find the relationship between \( C_2 \) and \( C_4 \) if \( C_2 = 2C_4. \)
2. Find the relationship between \( C_2 \) and \( C_3 \) if \( C_2 = C_3. \)
3. Find the relationship between \( C_2 \) and \( C_5 \) if \( C_2 = 2C_5. \)
4. Find the relationship between \( C_2 \) and \( C_6 \) if \( C_2 = C_6. \)

---

**Helping You Remember**

An important skill needed for working with logarithms is changing an equation between logarithmic and exponential form. Using the words here, express and logarithmic, describe an easy way to remember and apply the part of the definition of logarithm that says, \( \log_b a = y \) if and only if \( b^y = a. \) Sample answer: In these equations, it always helps to remember the number that is raised to a power. A logarithm is an exponent, so \( y, \) which is the log in the first equation, becomes the exponent in the second equation.
Exercises 85

Getting terms with like bases. Use familiar with exponential properties, students must be related to exponential logarithms. Because these properties are related to exponential properties, students must be familiar with exponential properties when multiplying or dividing terms with like bases. Use Exercises 85–90 to determine your students’ familiarity with multiplying and dividing monomials.

Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 10-1 and 10-2. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 10-1 and 10-2) is available on p. 623 of the Chapter 10 Resource Masters.

Maintain Your Skills

75. In the figure at the right, if \( y = \frac{2}{3}x \) and \( z = 3w \), then \( x = D \) 14. 20. 28. 35.

4 Assess

Open-Ended Assessment

Writing Have students write a step-by-step explanation of the procedure for solving a logarithmic equation such as \( \log_8 n = \frac{7}{3} \).

Getting Ready for Lesson 10-3

PREREQUISITE SKILL In Lesson 10-3, students will evaluate expressions using the properties of logarithms. Because these properties are related to exponential properties, students must be familiar with exponential properties when multiplying or dividing terms with like bases. Use Exercises 85–90 to determine your students’ familiarity with multiplying and dividing monomials.

Assessment Options

Practice Quiz 1 The quiz provides students with a brief review of the concepts and skills in Lessons 10-1 and 10-2. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

Quiz (Lessons 10-1 and 10-2) is available on p. 623 of the Chapter 10 Resource Masters.

Mixed Review

Simplify each expression. (Lesson 10-1)

76. \( x\sqrt{6} \cdot x\sqrt{6} \quad x^2\sqrt{6} \)

77. \( (\sqrt{6})^{\frac{1}{2}} \quad \sqrt{4} b^2 \)

Solve each equation. Check your solutions. (Lesson 9-6)

78. \( \frac{2x + 1}{x} - \frac{x + 1}{x - 4} = \frac{-20}{x^2 - 4x} \)

79. \( \frac{2a - 5}{a - 9} - \frac{a - 3}{3a + 2} = \frac{5}{3a^2 - 25a - 18} \)

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 6-5)

80. \( 9y^2 = 49 \quad \pm \frac{7}{3} \)

81. \( 2y^2 = 5y + 6 \quad \frac{5 \pm \sqrt{73}}{4} \)

Simplify each expression. (Lesson 9-2)

82. \( \frac{3}{2y} + \frac{4}{3y} - \frac{7}{5y} \quad 43 \quad 30y \)

83. \( \frac{x - 7}{x^2 - 9} - \frac{x - 3}{x^2 + 10x + 21} \)

84. BANKING Donna Bowers has $4000 she wants to save in the bank. A certificate of deposit (CD) earns 8% annual interest, while a regular savings account earns 3% annual interest. Ms. Bowers doesn’t want to tie up all her money in a CD, but she has decided she wants to earn $240 in interest for the year. How much money should she put in to each type of account? (Hint: Use Cramer’s Rule.) (Lesson 4-4) $2400, CD; $1600, savings

Getting Ready for the Next Lesson

PREREQUISITE SKILL Simplify. Assume that no variable equals zero. (To review multiplying and dividing monomials, see Lesson 5-1.)

85. \( x^4 \cdot x^6 \quad x^{10} \)

86. \( (y^3)^8 \quad y^{24} \)

87. \( (2a^2 b)^3 \quad 8a^6 b^3 \)

88. \( \frac{a^4 b^2}{a^3 b} \quad a^7 b^6 \)

89. \( \frac{x^5 y^2}{x^2 y^5} \quad \frac{x^3}{y^2} \)

90. \( \left( \frac{x^2 y}{a^3} \right)^0 \quad 1 \)

Practice Quiz 1

Lessons 10-1 and 10-2

1. Determine whether 5(1.2)^t represents exponential growth or decay. (Lesson 10-1) growth

2. Write an exponential function whose graph passes through (0, 2) and (2, 32). \( y = 2(4)^x \)

3. Write an equivalent logarithmic equation for \( 4^6 = 4096 \). (Lesson 10-2) \( \log_4 4096 = 6 \)

4. Write an equivalent exponential equation for \( \log_6 27 = \frac{3}{2} \). (Lesson 10-2) \( 3^\frac{3}{2} = 27 \)

Evaluate each expression. (Lesson 10-2)

5. \( \log_3 16 \quad \frac{4}{3} \)

6. \( \log_4 4^{15} \quad 15 \)

Solve each equation or inequality. Check your solutions. (Lessons 10-1 and 10-2)

7. \( 3^x = 3^3 - x \quad \frac{3}{5} \)

8. \( 3^{2n} \leq \frac{1}{9} \quad n \leq -1 \)

9. \( \log_2 (x + 6) > 5 \quad x > 26 \)

10. \( \log_5 (4x - 1) = \log_5 (3x + 2) \quad 3 \)
Modeling Real-World Data: Curve Fitting

We are often confronted with data for which we need to find an equation that best fits the information. We can find exponential and logarithmic functions of best fit using a TI-83 Plus graphing calculator.

Example

The population per square mile in the United States has changed dramatically over a period of years. The table shows the number of people per square mile for several years.

<table>
<thead>
<tr>
<th>Year</th>
<th>People per square mile</th>
<th>Year</th>
<th>People per square mile</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>4.5</td>
<td>1900</td>
<td>21.5</td>
</tr>
<tr>
<td>1800</td>
<td>6.1</td>
<td>1910</td>
<td>26.0</td>
</tr>
<tr>
<td>1810</td>
<td>4.3</td>
<td>1920</td>
<td>29.9</td>
</tr>
<tr>
<td>1820</td>
<td>5.5</td>
<td>1930</td>
<td>34.7</td>
</tr>
<tr>
<td>1830</td>
<td>7.4</td>
<td>1940</td>
<td>37.2</td>
</tr>
<tr>
<td>1840</td>
<td>9.8</td>
<td>1950</td>
<td>42.6</td>
</tr>
<tr>
<td>1850</td>
<td>7.9</td>
<td>1960</td>
<td>50.6</td>
</tr>
<tr>
<td>1860</td>
<td>10.6</td>
<td>1970</td>
<td>57.5</td>
</tr>
<tr>
<td>1870</td>
<td>10.9</td>
<td>1980</td>
<td>64.0</td>
</tr>
<tr>
<td>1880</td>
<td>14.2</td>
<td>1990</td>
<td>70.3</td>
</tr>
<tr>
<td>1890</td>
<td>17.8</td>
<td>2000</td>
<td>80.0</td>
</tr>
</tbody>
</table>

Source: Northeast-Midwest Institute

a. Use a graphing calculator to enter the data and draw a scatter plot that shows how the number of people per square mile is related to the year.

**Step 1** Enter the year into L1 and the people per square mile into L2.

**KEYSTROKES:** See pages 87 and 88 to review how to enter lists.

Be sure to clear the Y= list. Use the \( \downarrow \) key to move the cursor from L1 to L2.

**Step 2** Draw the scatter plot.

**KEYSTROKES:** See pages 87 and 88 to review how to graph a scatter plot.

Make sure that Plot 1 is on, the scatter plot is chosen, Xlist is L1, and Ylist is L2. Use the viewing window \([1780, 2020]\) with a scale factor of 10 by \([0, 115]\) with a scale factor of 5.

We see from the graph that the equation that best fits the data is a curve. Based on the shape of the curve, try an exponential model.

**Step 3** To determine the exponential equation that best fits the data, use the exponential regression feature of the calculator.

**KEYSTROKES:** \[ 8 \] [STAT] \[ 0 \] [2nd] [L1] \[ \downarrow \] [2nd] [L2] [ENTER]

The equation is \( y = 1.835122 \times 10^{-11}(1.014700091)^x \).

(continued on the next page)
In Exercise 2, make sure students can explain why their equation of best fit is a good choice. In Exercise 3, students’ answers may vary slightly. When you discuss Exercise 6, you may want to ask for any ideas students have about how to use the calculator to judge the relative merits of various models (quadratic, cubic, quartic, and exponential).

**Assess**

The calculator also reports an $r$ value of 0.991887235. Recall that this number is a correlation coefficient that indicates how well the equation fits the data. A perfect fit would be $r = 1$. Therefore, we can conclude that this equation is a pretty good fit for the data.

To check this equation visually, overlap the graph of the equation with the scatter plot.

**KEYSTROKES:** Y = VARS 5 [GRAPH]

**b. If this trend continues, what will be the population per square mile in 2010?**

To determine the population per square mile in 2010, from the graphics screen, find the value of $y$ when $x = 2010$.

**KEYSTROKES:** 2nd [CALC] 1 2010 [ENTER]

The calculator returns a value of approximately 100.6. If this trend continues, in 2010, there will be approximately 100.6 people per square mile.

**Exercises**

In 1985, Erika received $30 from her aunt and uncle for her seventh birthday. Her father deposited it into a bank account for her. Both Erika and her father forgot about the money and made no further deposits or withdrawals. The table shows the account balance for several years.

1. Use a graphing calculator to draw a scatter plot for the data.
2. Calculate and graph the curve of best fit that shows how the elapsed time is related to the balance. Use ExpReg for this exercise. See margin.
3. Write the equation of best fit. $y = 29.99908551(1.06500135)^x$
4. Write a sentence that describes the fit of the graph to the data.
5. Based on the graph, estimate the balance in 41 years. Check this using the CALC value. After 41 years she will have approximately $397.
6. Do you think there are any other types of equations that would be good models for these data? Why or why not? A quadratic equation might be a good model for this example because the shape is close to a portion of a parabola.
What You'll Learn

- Simplify and evaluate expressions using the properties of logarithms.
- Solve logarithmic equations using the properties of logarithms.

How are the properties of exponents and logarithms related?

In Lesson 5-1, you learned that the product of powers is the sum of their exponents.

\[ 9 \cdot 81 = 3^2 \cdot 3^4 \text{ or } 3^2 + 4 \]

In Lesson 10-2, you learned that logarithms are exponents, so you might expect that a similar property applies to logarithms. Let’s consider a specific case. Does \( \log_3 (9 \cdot 81) = \log_3 9 + \log_3 81 \)?

\[
\begin{align*}
\log_3 (9 \cdot 81) &= \log_3 (3^2 \cdot 3^4) \\
&= \log_3 (3^2 + 4) \\
&= 2 + 4 \text{ or } 6 \\
&= \text{Inverse property of exponents and logarithms}
\end{align*}
\]

\[
\begin{align*}
\log_3 9 + \log_3 81 &= \log_3 3^2 + \log_3 3^4 \\
&= 2 + 4 \text{ or } 6 \\
&= \text{Inverse property of exponents and logarithms}
\end{align*}
\]

So, \( \log_3 (9 \cdot 81) = \log_3 9 + \log_3 81 \).

Properties of Logarithms

Since logarithms are exponents, the properties of logarithms can be derived from the properties of exponents. The example above and other similar examples suggest the following property of logarithms.

**Key Concept**

**Product Property of Logarithms**

- **Words** The logarithm of a product is the sum of the logarithms of its factors.
- **Symbols** For all positive numbers \( m, n, \) and \( b \), where \( b \neq 1 \),
  \[
  \log_b mn = \log_b m + \log_b n.
  \]
- **Example** \( \log_3 (4)(7) = \log_3 4 + \log_3 7 \)

To show that this property is true, let \( b^x = m \) and \( b^y = n \). Then, using the definition of logarithm, \( x = \log_b m \) and \( y = \log_b n \).

\[
\begin{align*}
b^x \cdot b^y &= mn \\
b^x + y &= mn \\
\log_b b^x y &= \log_b mn \\
\log_b x + y &= \log_b mn \\
\text{Product of Powers} \\
\text{Property of Equality for Logarithmic Functions}
\end{align*}
\]

\[
\begin{align*}
x + y &= \log_b mn \\
\text{Inverse Property of Exponents and Logarithms}
\end{align*}
\]

\[
\begin{align*}
\log_b m + \log_b n &= \log_b mn \\
\text{Replace } x \text{ with } \log_b m \text{ and } y \text{ with } \log_b n.
\end{align*}
\]

You can use the Product Property of Logarithms to approximate logarithmic expressions.
LOGARITHMS

PROPERTIES OF LOGARITHMS

**Teaching Tip** When discussing the Product Property of Logarithms, point out that the logarithms used in the example (log₂ (4)(7) = log₂ 4 + log₂ 7) show the property applies to all logarithms and not just those that can be simplified. Be sure students did not get this impression from the earlier example where it was shown that log₃ (9 ∙ 81) = log₃ 9 + log₃ 81.

1. Use log₅ 2 ≈ 0.4307 to approximate the value of log₅ 250. 3.4307

**Teaching Tip** Some students may wonder how the approximation for log₂ 3 was determined since on most calculators the log button calculates only logarithms of base 10. State that log₂ 3 = log₁₀ 3 / log₁₀ 2, which can be evaluated using a calculator. Stress that this procedure will be formally discussed in Lesson 10-4.

2. Use log₆ 8 ≈ 1.1606 and log₆ 32 ≈ 1.9345 to approximate the value of log₆ 4. 0.7737

3. **SOUND** The sound made by a lawnmower has a relative intensity of 10⁹ or 90 decibels. Would the sound of ten lawnmowers running at that same intensity be ten times as loud or 900 decibels? Explain your reasoning. No; the sound of ten lawnmowers is perceived to be only 10 decibels louder than the sound of one lawnmower, or 100 decibels.

**Example 1 Use the Product Property**

Use log₂ 3 ≈ 1.5850 to approximate the value of log₂ 48.

\[
\log₂ 48 = \log₂ (2^4 \cdot 3) \\
= \log₂ 2^4 + \log₂ 3 \\
= 4 + \log₂ 3 \\
= 4 + 1.5850 \approx 5.5850
\]

Thus, log₂ 48 is approximately 5.5850.

Recall that the quotient of powers is found by subtracting exponents. The property for the logarithm of a quotient is similar.

**Key Concept Quotient Property of Logarithms**

- **Words** The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- **Symbols** \( \log_b \frac{m}{n} = \log_b m - \log_b n \).

**Example 2 Use the Quotient Property**

Use log₅ 5 ≈ 1.4650 and log₅ 20 ≈ 2.7268 to approximate log₅ 4.

\[
\log₅ 4 = \log₅ \frac{20}{5} \\
= \log₅ 20 - \log₅ 5 \\
= 2.7268 - 1.4650 \approx 1.2618
\]

Thus, log₅ 4 is approximately 1.2618.

**Example 3 Use Properties of Logarithms**

**SOUND** The loudness \( L \) of a sound in decibels is given by \( L = 10 \log_{10} R \), where \( R \) is the sound’s relative intensity. Suppose one person talks with a relative intensity of 10⁶ or 60 decibels. Would the sound of ten people each talking at that same intensity be ten times as loud or 600 decibels? Explain your reasoning.

Let \( L_1 \) be the loudness of one person talking. \( L_1 = 10 \log_{10} 10^6 \)

Let \( L_2 \) be the loudness of ten people talking. \( L_2 = 10 \log_{10} (10^6 \cdot 10^6) \)

Then the increase in loudness is \( L_2 - L_1 \).

\[
L_2 - L_1 = 10 \log_{10} (10^6 \cdot 10^6) - 10 \log_{10} 10^6 \\
= 10(\log_{10} 10^6 + \log_{10} 10^6) - 10 \log_{10} 10^6 \\
= 10 \log_{10} (10^6 + 10^6) - 10 \log_{10} 10^6 \\
= 10 \log_{10} 10 - 1 \log_{10} 10^6 \\
= 10(1) or 10
\]

The sound of two people talking is perceived by the human ear to be only about 10 decibels louder than the sound of one person talking, or 70 decibels.

**Unlocking Misconceptions**

**Power Property** After you have discussed the Power Property of Logarithms on p. 543, clarify that the property works for logarithms because they are equivalent to exponents. Stress that students should not read a statement such as \( \log₂ 5^3 = 3 \log₂ 5 \) and conclude that \( 5^3 = 3 \times 5 \).
Given \( \log_5 6 \approx 1.1133 \), approximate the value of \( \log_5 216 \).

\[ \log_5 216 = 3 \times \log_5 6 \quad \text{(Power Property)} \]

You will show that this property is true in Exercise 50.

**Example 4**  
**Power Property of Logarithms**

Given \( \log_4 36 \approx 1.2925 \), approximate the value of \( \log_4 36 \).

\[ \log_4 36 = \frac{3}{2} \log_4 4 \quad \text{(Power Property)} \]

Solve each equation.

**Example 5**  
**Solve Equations Using Properties of Logarithms**

a. \( 3 \log_5 x - \log_5 4 = \log_5 16 \)

\[ 3 \log_5 x - 1 = \log_5 16 \quad \text{(Quotient Property)} \]

\[ x^3 = 64 \quad \text{Taking the cube root of each side.} \]

\[ x = 4 \]

The solution is 4.

b. \( \log_4 x + \log_4 (x - 6) = 2 \)

\[ \log_4 x(x - 6) = 2 \quad \text{(Product Property)} \]

\[ x(x - 6) = 4^2 \quad \text{(Definition of logarithm)} \]

\[ x^2 - 6x - 16 = 0 \quad \text{Subtract 16 from each side.} \]

\[ x = -2 \quad \text{or} \quad x = 8 \quad \text{Zero Product Property} \]

\[ x = 8 \quad \text{or} \quad x = -2 \quad \text{Solve each equation.} \]

**CHECK** Substitute each value into the original equation.

\[ \log_4 8 + \log_4 (8 - 6) = 2 \quad \frac{2}{2} + 2 = 2 \]

\[ \log_4 2 + \log_4 (-2 - 6) = 2 \quad \log_4 (-2) + \log_4 (-8) = 2 \]

\[ \log_4 (8 - 2) = 2 \quad \log_4 16 = 2 \quad 2 = 2 \quad \text{Undefined, -2 is an extraneous solution and must be eliminated.} \]

The only solution is 8.
1. Name the properties that are used to derive the properties of logarithms.
2. OPEN ENDED Write an expression that can be simplified by using two or more properties of logarithms. Then simplify it.
3. FIND THE ERROR Umeko and Clemente are simplifying $\log_7 6 + \log_7 3 - \log_7 2$.

### Umeko
- $\log_7 6 + \log_7 3 - \log_7 2 = \log_7 18 - \log_7 2 = \log_7 9$

### Clemente
- $\log_7 6 + \log_7 3 - \log_7 2 = \log_7 9 - \log_7 2 = \log_7 7$ or $1$

Who is correct? Explain your reasoning. Umeko; see margin for explanation.

### Guided Practice

#### GUIDED PRACTICE KEY

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Solve each equation. Check your solutions.

7. $\log_3 42 - \log_3 n = \log_3 7$ \hspace{1cm} 8. $\log_2 3x + \log_2 5 = \log_2 30$ \hspace{1cm} 9. $2 \log_3 x = \log_3 9$ \hspace{1cm} 10. $\log_{10} a + \log_{10} (a + 21) = 2$

### Application

**MEDICINE** For Exercises 11 and 12, use the following information.
The pH of a person’s blood is given by $pH = 6.1 + \log_{10} \frac{B}{C}$, where $B$ is the concentration of bicarbonate, which is a base, in the blood and $C$ is the concentration of carbonic acid in the blood.

11. $pH = 6.1 + \log_{10} \frac{B}{C}$

12. Most people have a blood pH of 7.4. What is the approximate ratio of bicarbonate to carbonic acid for blood with this pH? 20:1

### Extra Practice

See page 850.

### Homework Help

Use $\log_2 2 \approx 0.3407$ and $\log_3 3 \approx 0.6826$ to approximate the value of each expression.

13. $\log_5 9$ \hspace{1cm} 14. $\log_5 8$ \hspace{1cm} 15. $\log_5 \frac{2}{3}$ \hspace{1cm} 16. $\log_5 \frac{2}{3}$ \hspace{1cm} 17. $\log_5 50$ \hspace{1cm} 18. $\log_5 30$ \hspace{1cm} 19. $\log_5 0.5$ \hspace{1cm} 20. $\log_5 \frac{10}{9}$ \hspace{1cm} 21. $\log_2 4$\hspace{1cm} 22. $\log_4 a + \log_4 9 = \log_4 27$ \hspace{1cm} 23. $\log_{10} 16 - \log_{10} 2$ \hspace{1cm} 24. $\log_2 24 - \log_2 (y + 5) = \log_2 8$ \hspace{1cm} 25. $\log_2 n = \frac{1}{4} \log_2 16 + \frac{1}{2} \log_2 49$ \hspace{1cm} 26. $2 \log_{10} 6 - \frac{1}{3} \log_{10} 27 = \log_{10} x$ \hspace{1cm} 27. $\log_{10} a + \log_{10} (a + 2) = \log_{10} 2$ \hspace{1cm} 28. $\log_8 (a^2 + 2) + \log_8 2 = 2$ \hspace{1cm} 29. $\log_2 (12b - 21) - \log_2 (b^2 - 3) = 2$ \hspace{1cm} 30. $\log_5 (y + 2) - \log_5 (y - 2) = 1$ \hspace{1cm} 31. $\log_3 0.1 + 2 \log_3 x = \log_3 2 + \log_3 5$ \hspace{1cm} 32. $\log_5 64 - \log_5 \frac{8}{3} + \log_2 2 = \log_5 4p$

### Answer

3. Clemente incorrectly applied the product and quotient properties of logarithms.

$log_7 6 + log_7 3 = log_7 (6 \cdot 3)$ or $log_7 18$ \hspace{1cm} Product Property of Logarithms

$log_7 18 - log_7 2 = log_7 (18 \div 2)$ or $log_7 9$ \hspace{1cm} Quotient Property of Logarithms
35. False; \( \log_b (2^2 + 2^3) = \log_2 12 \), \( \log_2 2^2 + \log_2 2^3 = 2 + 3 = 5 \), and \( \log_2 12 \neq 5 \) since \( 2^5 = 32 \neq 12 \).

Solve for \( n \). 34. \( \frac{1}{2}(x - 1) \)

\[ 33. \log_9 4 \cdot n - 2 \log_9 x = \log_9 x^2 \frac{1}{4} \]

34. \( \log_9 8 + 3 \log_9 n = 3 \log_9 (x - 1) \)

CRITICAL THINKING Tell whether each statement is true or false. If true, show that it is true. If false, give a counterexample.

35. For all positive numbers \( m \), \( n \), and \( b \), where \( b \neq 1 \), \( \log_b (m + n) = \log_b m + \log_b n \).

36. For all positive numbers \( m \), \( n \), and \( b \), where \( b \neq 1 \), \( \log_b x + m \log_b x = (n + m) \log_b x \).

See pp. 573A–573D.

37. EARTHQUAKES The great Alaskan earthquake in 1964 was about 100 times more intense than the Loma Prieta earthquake in 1989. Find the difference in the Richter scale magnitudes of the earthquakes. 2

BIOLOGY For Exercises 38–40, use the following information.

The energy \( E \) (in kilocalories per gram molecule) needed to transport a substance from the outside to the inside of a living cell is given by \( E = 1.4 \log_{10} C_2 - \log_{10} C_1 \), where \( C_1 \) is the concentration of the outside substance and \( C_2 \) is the concentration inside the cell.

38. Express the value of \( E \) as one logarithm.

\[ E = 1.4 \log \frac{C_2}{C_1} \]

39. Suppose the concentration of a substance inside the cell is twice the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside? (Use \( \log_{10} 2 = 0.3010 \).

40. Suppose the concentration of a substance inside the cell is four times the concentration outside the cell. How much energy is needed to transport the substance from the outside of the cell to the inside?

SOUND For Exercises 41–43, use the formula for the loudness of sound in Example 3 on page 542. Use \( \log_{10} 2 = 0.3010 \) and \( \log_{10} 3 = 0.47712 \).

41. A certain sound has a relative intensity of \( R \). By how many decibels does the sound increase when the intensity is doubled? 3

42. A certain sound has a relative intensity of \( R \). By how many decibels does the sound decrease when the intensity is halved? 3

43. A stadium containing 10,000 cheering people can produce a crowd noise of about 90 decibles. If every one cheers with the same relative intensity, how much noise, in decibels, is a crowd of 30,000 people capable of producing? Explain your reasoning.

About 95 decibels; see margin for explanation.

STAR LIGHT For Exercises 44–46, use the following information.

The brightness, or apparent magnitude, \( m \) of a star or planet is given by the formula

\[ m = 6 - 2.5 \log_{10} \frac{L}{L_0} \]

where \( L \) is the amount of light coming to Earth from the star or planet and \( L_0 \) is the amount of light from a sixth magnitude star.

44. Find the difference in the magnitudes of Sirius and the crescent moon.

5

45. Find the difference in the magnitudes of Saturn and Neptune.

7.5

46. RESEARCH Use the Internet or other reference to find the magnitude of the dimmest stars that we can now see with ground-based telescopes. about 22

More About... The Greek astronomer Hipparchus made the first known catalog of stars. He listed the brightness of each star on a scale of 1 to 6, the brightest being 1. With no telescope, he could only see stars as dim as the 6th magnitude. Source: NASA

**Answer**

43. \( L = 10 \log_{10} R \), where \( L \) is the loudness of the sound in decibels and \( R \) is the relative intensity of the sound. Since the crowd increased by a factor of 3, we assume that the intensity also increases by a factor of 3. Thus, we need to find the loudness of \( 3R \).

\[ L = 10 \log_{10} 3R; L = 10(\log_{10} 3 + \log_{10} R) \]

\[ L = 10 \log_{10} 3 + 10 \log_{10} R; \]

\[ L \approx 10(0.4771) + 90; L = 4.771 + 90 \text{ or about 95} \]
47. **CRITICAL THINKING** Use the properties of exponents to prove the Quotient Property of Logarithms. See margin.

48. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See pp. 573A–573D.

**How are the properties of exponents and logarithms related?**

Include the following in your answer:

- examples like the one shown at the beginning of the lesson illustrating the Quotient Property and Power Property of Logarithms, and

- an explanation of the similarity between one property of exponents and its related property of logarithms.

49. Simplify $2 \log_5 12 - \log_5 8 - 2 \log_5 3$.  
   **A** $\log_5 2$  **B** $\log_5 3$  **C** $\log_5 0.5$  **D** $1$

50. **SHORT RESPONSE** Show that $\log_b m^n = p \log_b m$ for any real number $p$ and positive number $m$ and $b$, where $b \neq 1$. See margin.

---

**Getting Ready for**

**Lesson 10-4**

**PREREQUISITE SKILL** Students will use common logarithms to solve exponential equations and inequalities in Lesson 10-4. The solution techniques involve using the skills they learned when solving logarithmic equations and inequalities. Use Exercises 63–66 to determine your students’ familiarity with solving logarithmic equations and inequalities.

**Assessment Options**

**Quiz (Lesson 10-3)** is available on p. 623 of the Chapter 10 Resource Masters.

**Mid-Chapter Test (Lessons 10-1 through 10-3)** is available on p. 625 of the Chapter 10 Resource Masters.

---

**Answers**

47. Let $b^x = m$ and $b^y = n$. Then $\log_b m = x$ and $\log_b n = y$.

\[
\begin{align*}
\frac{b^x}{b^y} &= \frac{m}{n} \\
\frac{b^x - y}{b^x} &= \frac{m}{n} \\
\log_b (b^x - y) &= \log_b \frac{m}{n} \\
x - y &= \log_b \frac{m}{n} \\
\log_b m - \log_b n &= \log_b \frac{m}{n}
\end{align*}
\]

**Quotient Property**

**Property of Equality for Logarithmic Equations**

**Inverse Property of Exponents and Logarithms**

**Replace x with $\log_b m$ and y with $\log_b n$.**

---

50. Let $b^x = m$, then $\log_b m = x$.

\[
(b^x)^p = \log_b m^p
\]

**Product of Powers**

\[
\log_b b^p = \log_b m^p
\]

**Property of Equality for Logarithmic Equations**

\[
x^p = \log_b m^p
\]

**Inverse Property of Exponents and Logarithms**

\[
\log_b m^p = \log_b m^p
\]

**Replace x with $\log_b m$.**
10-4 Common Logarithms

**What You’ll Learn**
- Solve exponential equations and inequalities using common logarithms.
- Evaluate logarithmic expressions using the Change of Base Formula.

**Vocabulary**
- common logarithm
- Change of Base Formula

**Why is a logarithmic scale used to measure acidity?**

The pH level of a substance measures its acidity. A low pH indicates an acid solution while a high pH indicates a basic solution. The pH levels of some common substances are shown.

The pH level of a substance is given by $\text{pH} = -\log[H^+]$, where $H^+$ is the substance’s hydrogen ion concentration in moles per liter.

Another way of writing this formula is $\text{pH} = -\log[H^+]$.

**COMMON LOGARITHMS**

You have seen that the base 10 logarithm function, $y = \log_{10} x$, is used in many applications. Base 10 logarithms are called common logarithms. Common logarithms are usually written without the subscript 10.

$$\log_{10} x = \log x, \quad x > 0$$

Most calculators have a $\text{LOG}$ key for evaluating common logarithms.

**Example 1** Find Common Logarithms

Use a calculator to evaluate each expression to four decimal places.

a. $\log 3$  
   **KEYSTROKES:** LOG 3 ENTER $≈ 0.4771$

b. $\log 0.2$  
   **KEYSTROKES:** LOG 0.2 ENTER $≈ -0.6990$

Sometimes an application of logarithms requires that you use the inverse of logarithms, or exponentiation.

$$10^{\log x} = x$$

**Example 2** Solve Logarithmic Equations Using Exponentiation

**EARTHQUAKES**

The amount of energy $E$, in ergs, that an earthquake releases is related to its Richter scale magnitude $M$ by the equation $\log E = 11.8 + 1.5M$. The Chilean earthquake of 1960 measured 8.5 on the Richter scale. How much energy was released?

$\log E = 11.8 + 1.5M$  
Write the formula.

$\log E = 11.8 + 1.5(8.5)$  
Replace $M$ with 8.5.

$\log E = 24.55$  
Simplify.

$10^{\log E} = 10^{24.55}$  
Write each side using exponents and base 10.

$E = 10^{24.55}$  
Inverse Property of Exponents and Logarithms

$E = 3.55 \times 10^{24}$  
Use a calculator.

The amount of energy released by this earthquake was about $3.55 \times 10^{24}$ ergs.

**Resource Manager**

**Workbook and Reproducible Masters**

**Chapter 10 Resource Masters**
- Study Guide and Intervention, pp. 591–592
- Skills Practice, p. 593
- Practice, p. 594
- Reading to Learn Mathematics, p. 595
- Enrichment, p. 596

**Transparencies**
- 5-Minute Check Transparency 10-4
- Real-World Transparency 10
- Answer Key Transparencies

**Technology**
- Interactive Chalkboard
In-Class Examples

1. Use a calculator to evaluate each expression to four decimal places.
   a. \( \log 6 \approx 0.7782 \)
   b. \( \log 0.35 \approx -0.4559 \)

2. **Earthquake** Refer to Example 2. The San Fernando Valley earthquake of 1994 measured 6.6 on the Richter scale. How much energy did this earthquake release? about 5.01 \( \times 10^{21} \) ergs

   **Teaching Tip** After discussing In-Class Example 2, have students compare the Richter scale magnitudes of the Chilean and San Fernando Valley earthquakes. \( 8.5 \div 6.6 \approx 1.29 \); the Chilean magnitude was about 29% greater. Then have them compare the energy released by the Chilean earthquake to the energy released by the San Fernando Valley earthquake. \( 3.55 \times 10^{24} + 5.01 \times 10^{21} \approx 708.58 \); the Chilean earthquake released more than 6 times as much energy. Point out that these results demonstrate the nonlinear nature of the equation that models the amount of energy released.

3. Solve \( 5^x = 62 \). about 2.5643

4. Solve \( 2^{7x} > 3^{5x} - 3 \). \( \{x \mid x < 5.1415\} \)

---

**Example 3** Solve Exponential Equations Using Logarithms

Solve \( 3^x = 11 \).

1. \( 3^x = 11 \) (Original equation)
2. \( \log 3^x = \log 11 \) (Property of Equality for Logarithmic Functions)
3. \( x \log 3 = \log 11 \) (Power Property of Logarithms)
4. \( x = \frac{\log 11}{\log 3} \) (Divide each side by \( \log 3 \)).
5. \( x = \frac{2.39784}{0.4771} \) (Use a calculator).
6. \( x = 5.01 \) The solution is approximately 5.01.

**CHECK** You can check this answer using a calculator or by using estimation. Since \( 3^2 = 9 \) and \( 3^3 = 27 \), the value of \( x \) is between 2 and 3. In addition, the value of \( x \) should be closer to 2 than 3, since 11 is closer to 9 than 27. Thus, 2.1828 is a reasonable solution.

---

**Example 4** Solve Exponential Inequalities Using Logarithms

Solve \( 5^{3y} < 8^y - 1 \).

1. \( 5^{3y} < 8^y - 1 \) (Original inequality)
2. \( \log 5^{3y} < \log 8^y - 1 \) (Property of Inequality for Logarithmic Functions)
3. \( 3y \log 5 < (y - 1) \log 8 \) (Power Property of Logarithms)
4. \( 3y \log 5 < y \log 8 - \log 8 \) (Distributive Property)
5. \( y(3 \log 5 - \log 8) < -\log 8 \) (Subtract \( \log 8 \) from each side).
6. \( y < \frac{-\log 8}{3 \log 5 - \log 8} \) (Distributive Property)
7. \( y < \frac{-(0.9031)}{3(0.6990) - 0.9031} \) (Use a calculator).
8. \( y < -0.7564 \) (The solution set is \( \{y \mid y < -0.7564\} \)).

**CHECK** Test \( y = -1 \).

1. \( 5^{-3} < 8^{-1} \) (Original inequality)
2. \( 5^{-3} < 8^{-1} \) (Replace \( y \) with 1).  
3. \( 5^{-3} < 8^{-2} \) (Simplify).
4. \( \frac{1}{125} < \frac{1}{64} \) (Negative Exponent Property)

---

**CHANGE OF BASE FORMULA** The **Change of Base Formula** allows you to write equivalent logarithmic expressions that have different bases.

**Key Concept**

**Change of Base Formula**

- **Symbols**
  - For all positive numbers, \( a \), \( b \), and \( n \), where \( a \neq 1 \) and \( b \neq 1 \),
  - \( \log_a n = \frac{\log_b n}{\log_b a} \) (log base \( b \) of \( n \) in terms of \( b \))
  - \( \log_a b = \frac{\log_b a}{\log_b b} \) (log base \( b \) of \( a \))

- **Example**
  - \( \log_5 12 = \frac{\log_{10} 12}{\log_{10} 5} \)

---

**Unlocking Misconceptions**

**Change of Base** As you discuss the **Change of Base Formula**, point out that the base \( b \) that students are changing to does not have to be 10. Any base could be used; however, \( b \) is most commonly 10 because this allows for the logarithms to be evaluated with a calculator.
To prove this formula, let \( \log_b n = x \).

\[
\begin{align*}
\log_b a^t &= n & \text{Definition of logarithm} \\
\log_b a^t &= \log_b n & \\
x \log_b a &= \log_b n & \text{Property of Equality for Logarithms} \\
x &= \frac{\log_b n}{\log_b a} & \text{Divide each side by } \log_b a. \\
\log_b n &= \frac{\log_b n}{\log_b a} & \text{Replace } x \text{ with } \log_b n.
\end{align*}
\]

This formula makes it possible to evaluate a logarithmic expression of any base by translating the expression into one that involves common logarithms.

**Example 5** Change of Base Formula

Express \( \log_4 25 \) in terms of common logarithms. Then approximate its value to four decimal places.

\[
\log_4 25 = \frac{\log_{10} 25}{\log_{10} 4}
\]

Change of Base Formula

\[
= \frac{2.3219}{0.6020}
\]

Use a calculator.

The value of \( \log_4 25 \) is approximately 2.3219.

### Check for Understanding

#### Concept Check

1. Name the base used by the calculator \( \log \) key. What are these logarithms called? \( 10; \) common logarithms

2. **OPEN ENDED** Give an example of an exponential equation requiring the use of logarithms to solve. Then solve your equation.

3. Explain why you must use the Change of Base Formula to find the value of \( \log_5 7 \) on a calculator. A calculator is not programmed to find base 2 logarithms.

### Guided Practice

Use a calculator to evaluate each expression to four decimal places.

4. \( \log 4 \ 0.6021 \)

5. \( \log 23 \ 1.3617 \)

6. \( \log 0.5 \ -0.3010 \)

Solve each equation or inequality. Round to four decimal places.

7. \( 9^x = 45 \ 1.7325 \)

8. \( 4^b > 30 \ (b \ n > 0.4907) \)

9. \( 3.1^x - 3 = 9.42 \ 4.9824 \)

10. \( 11^x = 25.4 \ 0.8271 \)

11. \( 7^x - 2 = 5^y \ 11.5665 \)

12. \( 4^x - 3 \leq 3^y \)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

13. \( \log_7 5 \ 0.8271 \)

14. \( \log_3 42 \ \frac{2.4022}{3.1699} \)

15. \( \log_2 9 \ \frac{9}{2.1699} \)

16. **DIET** Sandra’s doctor has told her to avoid foods with a pH that is less than 4.5. What is the hydrogen ion concentration of foods Sandra is allowed to eat? Use the information at the beginning of the lesson. at least 0.00003 mole per liter

### Application

Use a calculator to evaluate each expression to four decimal places.

17. \( \log 5 \ 0.6990 \)

18. \( \log 12 \ 1.0792 \)

19. \( \log 7.2 \ 0.8573 \)

20. \( \log 2.3 \ 0.3617 \)

21. \( \log 0.8 \ -0.0969 \)

22. \( \log 0.03 \ -1.5229 \)

www.algebra2.com/extra_examples

### About the Exercises...

**Organization by Objective**

- **Common Logarithms:** 17–44, 51–55
- **Change of Base Formula:** 45–50

**Odd/Even Assignments**

Exercises 17–52 are structured so that students practice the same concepts whether they are assigned odd or even problems.

** Assignment Guide**

**Basic:** 17–41 odd, 45, 47, 51, 56, 59–77

**Average:** 17–51 odd, 56–77

**Advanced:** 18–52 even, 53–74 (optional: 72–77)
ACIDITY For Exercises 23–26, use the information at the beginning of the lesson to find the pH of each substance given its concentration in solution.

23. ammonia: $[H+] = 1 \times 10^{-11}$ mole per liter

24. vinegar: $[H+] = 6.3 \times 10^{-3}$ mole per liter

25. lemon juice: $[H+] = 7.9 \times 10^{-3}$ mole per liter

26. orange juice: $[H+] = 3.16 \times 10^{-4}$ mole per liter

Solve each equation or inequality. Round to four decimal places.

27. $6^x \geq 12$ (x ≥ 2.0860)

28. $5^x = 52$ (x = 2.5450)

29. $8^{x^2} < 124$ ($a^a < 1.1590$)

30. $3^x + 2 = 14.5$ (x = 4.039)

31. $3^{x^2} - 2 = 42.5$ (x = 3.462)

32. $2^{x^2} = 70$ (x = ±1.1909)

33. $8^{x^2} > 52 + 3$ (a^n > -0.1078)

34. $2^{x^2} - 3 = 3x$ (x = 1.0890)

35. $7^x + 2 = 135^2 - p$ (p = ±1.9830)

36. $4^{x^2} = 0.3869$

37. $4^{x^2} = 0.3869$

38. $4^{x^2} = 0.3869$

39. $4^{x^2} = 0.3869$

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78. $4^{x^2} = 0.3869$

79. $4^{x^2} = 0.3869$

80. $4^{x^2} = 0.3869$

81. $4^{x^2} = 0.3869$

82. $4^{x^2} = 0.3869$

83. $4^{x^2} = 0.3869$

84. $4^{x^2} = 0.3869$

85. $4^{x^2} = 0.3869$

86. $4^{x^2} = 0.3869$

87. $4^{x^2} = 0.3869$

88. $4^{x^2} = 0.3869$

89. $4^{x^2} = 0.3869$

90. $4^{x^2} = 0.3869$

91. $4^{x^2} = 0.3869$

92. $4^{x^2} = 0.3869$

93. $4^{x^2} = 0.3869$

94. $4^{x^2} = 0.3869$

95. $4^{x^2} = 0.3869$

96. $4^{x^2} = 0.3869$

97. $4^{x^2} = 0.3869$

98. $4^{x^2} = 0.3869$

99. $4^{x^2} = 0.3869$

100. $4^{x^2} = 0.3869$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

45. $\log_{10} 13 \approx 1.7095$

46. $\log_{10} 20 \approx 2.0000$

47. $\log_{10} 5 \approx 0.6990$

48. $\log_{10} 3 \approx 0.4771$

49. $\log_{10} 2 \approx 0.3010$

50. $\log_{10} 6 \approx 0.7781$

More About...

53. Sirius

As little as 0.9 milligram per liter of iron at a pH of 5.5 can cause fish to die. Source: Kentucky Water Watch

For Exercises 51 and 52, use the information presented at the beginning of the lesson.

51. POLLUTION The acidity of water determines the toxic effects of runoff into streams from industrial or agricultural areas. A pH range of 6.0 to 9.0 appears to provide protection for freshwater fish. What is this range in terms of the water’s hydrogen ion concentration? between 0.000000001 and 0.000001 mole per liter

52. BUILDING DESIGN The 1971 Sylmar earthquake in Los Angeles had a Richter scale magnitude of 6.3. Suppose an architect has designed a building strong enough to withstand 50 earthquakes at intense as the Sylmar quake. Find the magnitude of the strongest quake this building is designed to withstand. 8

ASTRONOMY For Exercises 53–55, use the following information.

Some stars appear bright only because they are very close to us. Absolute magnitude M is a measure of how bright a star would appear if it were 10 parsecs, about 32 light years, away from Earth. A lower magnitude indicates a brighter star. Absolute magnitude is given by $M = m + 5 - 5 \log d$, where $d$ is the star’s distance from Earth measured in parsecs and $m$ is its apparent magnitude.

Sirius and Vega are two of the brightest stars in Earth’s sky. The apparent magnitude of Sirius is $-1.44$ and of Vega is 0.03. Which star appears brighter?

Sirius is 2.64 parsecs from Earth while Vega is 7.76 parsecs from Earth. Find the absolute magnitude of each star.

Sirius: 1.45, Vega: 0.58

55. Which star is actually brighter? That is, which has a lower absolute magnitude?

Vega

56. CRITICAL THINKING

a. Without using a calculator, find the value of $\log_{10} 5$ and $\log_{10} 2$

b. Without using a calculator, find the value of $\log_{10} 27$ and $\log_{10} 9$

c. Make and prove a conjecture as to the relationship between $\log_{10} b$ and $\log_{10} a$.

See margin.

Reading to Learn Mathematics, p. 593

ELL

Pre-Activity Why is a logarithmic scale used to measure acidity?

Read the introduction to Lesson 10-4 on page 597 in your textbook. What substance is more acidic, milk or tomatoes?

Homework Help

For Exercise See Examples...

17–22 1

23–44 3, 4

53–57 5

55–55 2

Extra Practice

See page 850.

550 Chapter 10 Exponential and Logarithmic Relations

Enrichment, p. 596

The Slide Rule

Before the invention of electronic calculators, computations were often performed on a slide rule. A slide rule is based on the idea of logarithms. It has two movable rods labeled with C and D scales. Each of the scales is logarithmic.
Getting Ready for the Next Lesson

**Maintain Your Skills**

Use \( \log_2 2 = 0.3562 \) and \( \log_3 3 = 0.5646 \) to approximate the value of each expression. (Lesson 10-3)

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log 10^3 )</td>
<td>( \log 10^2 )</td>
</tr>
</tbody>
</table>

Solve each equation or inequality. Check your solutions. (Lesson 10-2)

65. \( \log_x r = 3 \)
66. \( \log_b z \leq -2 \)
67. \( \log_3 (4x - 5) = 5 \)
68. Use synthetic substitution to find \( f(-2) \) for \( f(x) = x^3 + 6x - 2 \). (Lesson 7-4)

Factor completely. If the polynomial is not factorable, write prime. (Lesson 5-4)

69. \( 2x^2 + 2x - 8 \)
70. \( 42y^2 - 35y + 18y - 15 \)
71. \( 13x^2 + 3x^2 + 4k \)

**PREREQUISITE SKILLS**

Write an equivalent exponential equation. (Lesson 10-2)

72. \( \log_2 3 = x \) \( 2^x = 3 \)
73. \( \log_5 x = 2 \) \( 3^x = x \)
74. \( \log_5 125 = 3 \) \( 5^3 = 125 \)

Write an equivalent logarithmic equation. (Lesson 10-2)

75. \( 5^x = 45 \) \( \log_5 45 = x \)
76. \( 7^3 = x \) \( \log_7 x = 3 \)
77. \( b^y = x \) \( \log_b x = y \)

**Open-Ended Assessment**

**Writing**

Ask students to explain in writing what it means to use the Change of Base Formula. They should include comments about why this formula is useful.

**Getting Ready for Lesson 10-5**

**PREREQUISITE SKILL**

In Lesson 10-5, students will solve exponential equations and inequalities using natural logarithms and the skills they learned solving common logarithmic equations and inequalities. Students should be confident when converting between exponential and logarithmic equations before proceeding. Use Exercises 72–77 to determine your students’ familiarity with converting between exponential and logarithmic equations.

**Answers**

56c. conjecture: \( \log_a b = \frac{1}{\log_b a} \)

- **Proof:**
  \[
  \log_a b = \frac{1}{\log_b a}
  \]

59. Comparisons between substances of different acidities are more easily distinguished on a logarithmic scale. Answers should include the following.

- **Sample answer:**
  - Tomatoes: \( 6.3 \times 10^{-5} \) mole per liter
  - Milk: \( 3.98 \times 10^{-7} \) mole per liter
  - Eggs: \( 1.58 \times 10^{-8} \) mole per liter

- Those measurements correspond to pH measurements of 5 and 4, indicating a weak acid and a stronger acid. On the logarithmic scale we can see the difference in these acids, whereas on a normal scale, these hydrogen ion concentrations would appear nearly the same. For someone who has to watch the acidity of the foods they eat, this could be the difference between an enjoyable meal and heartburn.
A Follow-Up of Lesson 10-4

Getting Started

Using Parentheses In Step 1 of Example 1, remind students that they must also use parentheses around the fraction $\frac{1}{2}$.

Teach

• Before discussing Example 1, use a simple equation such as $2x = 6$ to show students how the equation can be solved by graphing. Graph the equations $y = 2x$ and $y = 6$ and then identify the point of intersection of the graphs.

• Ask students why it is necessary in Step 1 to enter the equations using parentheses around the exponents.

• Have students substitute the solution to Example 1 into the original equation to verify that it is correct.

• In Example 2, make sure students understand why the equations must be rewritten using the Change of Base Formula.

• Students can find the solution set for Example 2 without using the shading options. Simply have them use the intersect feature, noting that the graph of $Y_1$ intersects or is above the graph of $Y_2$ at and to the right of $x = 0.5$.

Example 1

Solve $2^{3x} - 9 = (\frac{1}{2})^{x - 3}$ by graphing.

Step 1 Graph each side of the equation.

• Graph each side of the equation as a separate function. Enter $2^{3x} - 9$ as $Y_1$. Enter $(\frac{1}{2})^{x - 3}$ as $Y_2$. Be sure to include the added parentheses around each exponent. Then graph the two equations.

KEYSTROKES: See pages 87 and 88 to review graphing equations.

Step 2 Use the intersect feature.

• You can use the intersect feature on the CALC menu to approximate the ordered pair of the point at which the curves cross.

KEYSTROKES: See page 115 to review how to use the intersect feature.

The TI-83 Plus has $y = \log_{10} x$ as a built-in function. Enter $\text{Y= LOG} [X,T,\theta,n] \text{ GRAPH}$ to view this graph. To graph logarithmic functions with bases other than 10, you must use the Change of Base Formula,

$$\log_a n = \frac{\log_b n}{\log_b a}.$$ 

For example, $\log_3 x = \frac{\log_{10} x}{\log_{10} 3}$, so to graph $y = \log_3 x$ you must enter $\text{LOG} [X,T,\theta,n] \div \text{LOG} 3$ as $Y_1$. 

www.algebra2.com/other_calculator_keystrokes
Example 2
Solve \( \log_2 2x \geq \log_2 3x \) by graphing.

**Step 1** Rewrite the problem as a system of common logarithmic inequalities.
- The first inequality is \( \log_2 2x \geq y \) or \( y \leq \log_2 3x \). The second inequality is \( y = \frac{\log 2x}{\log 2} \).
- Use the Change of Base Formula to create equations that can be entered into the calculator.

\[
\log_2 2x = \frac{\log 2x}{\log 2} \quad \log_2 3x = \frac{\log 3x}{\log 2}
\]

Thus, the two inequalities are \( y \leq \frac{\log 2x}{\log 2} \) and \( y = \frac{\log 3x}{\log 2} \).

**Step 2** Enter the first inequality.
- Enter \( \frac{\log 2x}{\log 2} \) as \( \text{Y1} \). Since the inequality includes less than, shade below the curve.

**Step 3** Enter the second inequality.
- Enter \( \frac{\log 3x}{\log 2} \) as \( \text{Y2} \). Since the inequality includes greater than, shade above the curve.

**Step 4** Graph the inequalities.

Exercises
Solve each equation or inequality by graphing.

1. \( 3.5^x + 2 = 1.75^x + 3 \quad -1.2 \)
2. \( -3^x + 4 = -0.52^x + 3 \quad -2.6 \)
3. \( 6^x - x - 4 = -0.25^x - 2.5 \quad 1.8 \)
4. \( 3^x - 4 = 5^x \quad 2 \)
5. \( \log_3 3x = \log_3 (2x + 2) \quad 0.7 \)
6. \( 2^x - 2 \geq 0.5^x - 3 \quad x \geq 2.5 \)
7. \( x \geq 6 \)
8. \( 5^x + 3 \leq 2^x + 4 \quad x \leq -2.24 \)
9. \( \log_2 2x \leq \log_4 (x + 3) \quad 0 < x \leq 1 \)
Base e and Natural Logarithms

What You’ll Learn
• Evaluate expressions involving the natural base and natural logarithms.
• Solve exponential equations and inequalities using natural logarithms.

Vocabulary
• natural base, e
• natural base exponential function
• natural logarithm
• natural logarithmic function

How is the natural base e used in banking?
Suppose a bank compounds interest on accounts continuously, that is, with no waiting time between interest payments. In order to develop an equation to determine continuously compounded interest, examine what happens to the value A of an account for increasingly larger numbers of compounding periods n. Use a principal P of $1, an interest rate r of 100% or 1, and time t of 1 year.

BASE e AND NATURAL LOGARITHMS
In the table above, as n increases, the expression \((1 + \frac{1}{n})^n\) or \((1 + \frac{1}{n})^n\) approaches the irrational number 2.71828... . This number is referred to as the natural base, e.

An exponential function with base e is called a natural base exponential function. The graph of \(y = e^x\) is shown at the right. Natural base exponential functions are used extensively in science to model quantities that grow and decay continuously.

Most calculators have an e^x function for evaluating natural base expressions.

Example 1 Evaluate Natural Base Expressions
Use a calculator to evaluate each expression to four decimal places.

a. \(e^2\)KEYSTROKES: 2nd \(e^x\) 2 ENTER 7.389056099 about 7.3891

b. \(e^{-1.3}\)KEYSTROKES: 2nd \(e^x\) – 1.3 ENTER .722531793 about 0.2725

The logarithm with base e is called the natural logarithm, sometimes denoted by \(\log_e x\), but more often abbreviated \(\ln x\). The natural logarithmic function, \(y = \ln x\), is the inverse of the natural base exponential function, \(y = e^x\). The graph of these two functions shows that \(\ln 1 = 0\) and \(\ln e = 1\).
Most calculators have an LN key for evaluating natural logarithms.

**Example 1** Evaluate Natural Logarithmic Expressions

Use a calculator to evaluate each expression to four decimal places.

a. ln 4 KEystrokes: LN 4 ENTER 1.386294361 about 1.3863

b. ln 0.05 KEystrokes: LN 0.05 ENTER -2.995732274 about -2.9957

You can write an equivalent base e exponential equation for a natural logarithmic equation and vice versa by using the fact that ln x = logₐ x.

**Example 2** Write Equivalent Expressions

Write an equivalent exponential or logarithmic equation.

a. e⁵ = 5
   
   e⁵ = 5 → log₅ 5 = x

b. ln x ≈ 0.6931
   
   ln x ≈ 0.6931 → logₑ x ≈ 0.6931

Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to “undo” each other.

\[ e^{\ln x} = x \quad \ln e^x = x \]

**Example 3** Inverse Property of Base e and Natural Logarithms

Evaluate each expression.

a. \( e^{\ln 7} \)

   \[ e^{\ln 7} = 7 \]

b. \( \ln e^{2x + 3} \)

   \[ \ln e^{2x + 3} = 2x + 3 \]

**EQUATIONS AND INEQUALITIES WITH e AND ln** Equations and inequalities involving base e are easier to solve using natural logarithms than using common logarithms. All of the properties of logarithms that you have learned apply to natural logarithms as well.

**Example 4** Solve Base e Equations

Solve \( 5e^{-x} - 7 = 2 \).

\[
5e^{-x} - 7 = 2 \\
5e^{-x} = 9 \\
e^{-x} = \frac{9}{5} \\
\ln e^{-x} = \ln \frac{9}{5} \\
-x = \ln \frac{9}{5} \\
x = -\ln \frac{9}{5} \\
x \approx -0.5878
\]

The solution is about -0.5878.

**CHECK** You can check this value by substituting -0.5878 into the original equation or by finding the intersection of the graphs of \( y = 5e^{-x} - 7 \) and \( y = 2 \).

www.algebra2.com/extra_examples

Lesson 10-5 Base e and Natural Logarithms 555
Suppose you deposit $700 into an account paying 6% annual interest, compounded continuously.

a. What is the balance after 8 years? $1131.25

b. How long will it take for the balance in your account to reach at least $2000? At least 17.5 years

Solve each equation or inequality.

a. $\ln 3x = 0.5$ About 0.5496

b. $\ln (2x - 3) < 2.5$ $1.5 < x < 7.5912$

When interest is compounded continuously, the amount $A$ in an account after $t$ years is found using the formula $A = Pe^{rt}$, where $P$ is the amount of principal and $r$ is the annual interest rate.

Example 6  Solve Base e Inequalities

Suppose you deposit $1000 in an account paying 5% annual interest, compounded continuously.

a. What is the balance after 10 years? $1648.72

b. How long will it take for the balance in your account to reach at least $1500? At least 8.11 years

Example 7  Solve Natural Log Equations and Inequalities

Solve each equation or inequality.

a. $\ln 5x = 4$ $x = 10.9196$

b. $\ln (x - 1) > -2$ $x > 1.1353$

Kinesthetic  Using plastic coins and paper currency, have pairs of students begin with $10, choose an interest rate, and calculate how much they will have after 5, 10, 15, and 20 years. After each calculation, have students model the amount with their money to help them visualize the growth over time.
1. Name the base of natural logarithms. \textbf{the number } e \\
2. OPEN ENDED Give an example of an exponential equation that requires using natural logarithms instead of common logarithms to solve. \textbf{Sample answer: } \[ e^x = 8 \]
3. FIND THE ERROR Colby and Elsu are solving \( \ln 4x = 5 \).

<table>
<thead>
<tr>
<th>Colby</th>
<th>Elsu</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln 4x = 5 )</td>
<td>( \ln 4x = 5 )</td>
</tr>
<tr>
<td>( 10^{\ln 4x} = 10^5 )</td>
<td>( e^{\ln 4x} = e^5 )</td>
</tr>
<tr>
<td>( 4x = 100,000 )</td>
<td>( 4x = e^5 )</td>
</tr>
<tr>
<td>( x = 25,000 )</td>
<td>( x = \frac{e^5}{4} )</td>
</tr>
<tr>
<td>( x \approx 37,105.5 )</td>
<td>( x \approx 37.1055 )</td>
</tr>
</tbody>
</table>

Who is correct? Explain your reasoning.

Use a calculator to evaluate each expression to four decimal places.

- 4. \( e^6 \approx 403.4288 \)
- 5. \( e^{-3.4} \approx 0.0334 \)
- 6. \( \ln 1.2 \approx 0.1823 \)
- 7. \( \ln 0.1 \approx -2.3026 \)

Write an equivalent exponential or logarithmic equation.

- 8. \( e^2 = 4 \), \( x = \ln 4 \)
- 9. \( \ln 1 = 0 \), \( e^0 = 1 \)

Evaluate each expression.

- 10. \( e^{\ln 3} = 3 \)
- 11. \( \ln e^{5x} = 5x \)

Solve each equation or inequality.

- 12. \( e^x > 30 \), \( x > 3.4012 \)
- 13. \( 2e^x - 5 = 1 \), \( 1.0866 \)
- 14. \( 3 + e^{-2x} = 8 \), \( -0.8047 \)
- 15. \( \ln x < 6 \)
- 16. \( 2 \ln 3x + 1 = 5 \), \( 2.4630 \)
- 17. \( \ln x^2 = 9 \), \( \pm 90.0171 \)

ALTITUDE For Exercises 18 and 19, use the following information.

The altimeter in an airplane gives the altitude or height \( h \) (in feet) of a plane above sea level by measuring the outside air pressure \( P \) (in kilopascals). The height and air pressure are related by the model \( P = 101.3 e^{-\frac{h}{20}} \).

18. \( h = \frac{P}{20} \)

- 19. Find a formula for the height in terms of the outside air pressure.

- 20. Use the formula you found in Exercise 18 to approximate the height of a plane above sea level when the outside air pressure is 57 kilopascals. \textbf{about 15,066 ft}

Use a calculator to evaluate each expression to four decimal places.

- 21. \( e^5 \approx 148.4132 \)
- 22. \( e^{-1.2} \approx 0.3012 \)
- 23. \( e^{0.5} \approx 1.6487 \)
- 24. \( \ln 3 \approx 1.0986 \)
- 25. \( \ln 10 \approx 2.3026 \)
- 26. \( \ln 5.42 \approx 1.6901 \)
- 27. \( \ln 0.03 \approx -3.5066 \)

28. SAVINGS If you deposit $150 in a savings account paying 4% interest compounded continuously, how much money will you have after 5 years? Use the formula presented in Example 6. \textbf{\$183.21}

29. PHYSICS The equation \( \ln \frac{I_0}{I} = 0.014d \) relates the intensity of light at a depth of \( d \) centimeters of water \( I \) with the intensity in the atmosphere \( I_0 \). Find the depth of the water where the intensity of light is half the intensity of the light in the atmosphere. \textbf{about 49.5 cm}

About the Exercises...

Organization by Objective
- Base \( e \) and Natural Logarithms: 20–37
- Equations and Inequalities with \( e \) and \( \ln \): 38–61

Odd/Even Assignments
Exercises 20–53 are structured so that students practice the same concepts whether they are assigned odd or even problems.

Assignment Guide
Advanced: 20–52 even, 54–74 (optional: 75–80)
All: Practice Quiz 2 (1–5)
How long will it take the balance in Sarita’s account to reach $2000?

Investing

Use a calculator to evaluate each expression to four decimal places.

If Sarita deposits $1000 in an account paying 3.4% annual interest compounded annually, investors use the “Rule of 72.” Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $72/6$ or 12 years.

Source: www.datachartp.com

Money

To determine the doubling time on an account paying an interest rate $r$ that is compounded annually, investors use the “Rule of 72.” Thus, the amount of time needed for the money in an account paying 6% interest compounded annually to double is $72/6$ or 12 years.

Money

For Exercises 54–57, use the formula for continuously compounded interest.

In 2000, the world’s population was about 6 billion. If the world’s population continues to grow at a constant rate, the future population $P$, in billions, can be predicted by $P = 6e^{0.02t}$, where $t$ is the time in years since 2000. 58. about 7.33 billion

According to this model, how many more years will the world’s population remain at 18 billion or less? 59. about 55 yr

Some experts have estimated that the world’s food supply can support a population of, at most, 18 billion. According to this model, how many more years will the world’s population remain at 18 billion or less? 59. about 55 yr

Online Research Data Update What is the current world population? Visit www.algebra2.com/data_update to learn more.

RUMORS

For Exercises 60 and 61, use the following information.

The number of people who have heard a rumor can be approximated by $P = 1 + (P_0 - 1)e^{-0.3t}$, where $P$ is the total population, $S$ is the number of people who start the rumor, and $t$ is the time in minutes. Suppose two students start a rumor that will principal who will tell everyone out of school one hour early that day.

If 1600 students in the school, how many students will have heard the rumor after 10 minutes? about 32 students

61. How much time will pass before half of the students have heard the rumor? about 21 min

CRITICAL THINKING Determine whether the following statement is sometimes, always, or never true. Explain your reasoning. Always; see pp. 573A-573D.

For all positive numbers $x$ and $y$, $\log x/\log y = \log x - \log y$.
63. **WRITING IN MATH** Answer the question that was posed at the beginning of the lesson. See margin.

How is the natural base $e$ used in banking?

Include the following in your answer:
- an explanation of how to calculate the value of an account whose interest is compounded continuously, and
- an explanation of how to use natural logarithms to find when the account will have a specified value.

64. If $e^{x} \neq 1$ and $e^{x^2} = \frac{1}{\sqrt{2}}$, what is the value of $x$? **B**

A $-1.41$  B $-0.35$  C $1.00$  D $1.10$

65. **SHORT RESPONSE** The population of a certain country can be modeled by the equation $P(t) = 40(1.02)^t$, where $P$ is the population in millions and $t$ is the number of years since 1900. When will the population be 100 million, 200 million, and 400 million? What do you notice about these time periods? 1946, 1981, 2015; It takes between 34 and 35 years for the population to double.

### Maintain Your Skills

#### Mixed Review

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. **(Lesson 10-4)**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Approximate Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_6 68$</td>
<td>$3.0437$</td>
</tr>
<tr>
<td>$\log_4 6$</td>
<td>$0.4014$</td>
</tr>
<tr>
<td>$\log_{50} 23$</td>
<td>$0.8015$</td>
</tr>
</tbody>
</table>

Solve each equation. Check your solutions. **(Lesson 10-3)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_3 (a + 3) + \log_3 (a - 3) = \log_3 16$</td>
<td>$5$</td>
</tr>
<tr>
<td>$\log_{11} 2 + 2 \log_{11} x = \log_{11} 32$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

State whether each equation represents a direct, joint, or inverse variation. Then name the constant of variation. **(Lesson 9-4)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$nt = 4$</td>
<td>inverse, $4$</td>
</tr>
<tr>
<td>$\frac{a}{b} = c$</td>
<td>joint, $1$</td>
</tr>
<tr>
<td>$y = -7x$</td>
<td>direct, $-7$</td>
</tr>
</tbody>
</table>

74. **COMMUNICATION** A microphone is placed at the focus of a parabolic reflector to collect sounds for the television broadcast of a football game. The focus of the parabola that is the cross section of the reflector is 5 inches from the vertex. The latus rectum is 20 inches long. Assuming that the focus is at the origin and the parabola opens to the right, write the equation of the cross section. **(Lesson 8-2)**

$x = \frac{1}{20}y^2 - 5$

### Getting Ready for the Next Lesson

#### PREREQUISITE SKILL

Solve each equation or inequality. **(Lesson 10-1)**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^x = 10$</td>
<td>$3.32$</td>
</tr>
<tr>
<td>$5^x = 12$</td>
<td>$1.54$</td>
</tr>
<tr>
<td>$6^x = 13$</td>
<td>$1.43$</td>
</tr>
<tr>
<td>$2(1 + 0.1)x^2 = 50$</td>
<td>$323.49$</td>
</tr>
<tr>
<td>$10(1 + 0.25)x = 200$</td>
<td>$13.43$</td>
</tr>
<tr>
<td>$400(1 - 0.2)x = 50$</td>
<td>$9.32$</td>
</tr>
</tbody>
</table>

#### Practice Quiz 2

**Lessons 10-3 through 10-5**

1. Express $\log_{3} 5$ in terms of common logarithms. Then approximate its value to four decimal places. **(Lesson 10-4)**

   $\log_{4} 5 \approx 1.6160$

2. Write an equivalent exponential equation for $\ln 3x = 2$. **(Lesson 10-5)**

   $e^2 = 3x$

Solve each equation or inequality. **(Lesson 10-5)**

3. $\log_{3} (9x + 5) = 2 + \log_{2} (x^2 - 1)$ | $3$ |

   $2x - 3 > 5$ | $x > 5.3219$ |

   $5. 2e^x - 1 = 7$ | $1.3863$ |

**Lesson 10-5** Base $e$ and Natural Logarithms 559

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**Open-Ended Assessment**

**Speaking** Ask students to explain how evaluating expressions involving base $e$ and natural logarithms is similar to evaluating expressions involving common logarithms and base 10, and also how they differ.

**Getting Ready for Lesson 10-6**

**PREREQUISITE SKILL** Students will encounter exponential growth and decay problems in Lesson 10-6. They will be required to solve exponential equations and inequalities. Use Exercises 75–80 to determine your students’ familiarity with solving exponential equations and inequalities.

**Assessment Options**

**Practice Quiz 2** The quiz provides students with a brief review of the concepts and skills in Lessons 10-3 through 10-5. Lesson numbers are given to the right of the exercises or instruction lines so students can review concepts not yet mastered.

**Quiz (Lessons 10-4 and 10-5)** is available on p. 624 of the Chapter 10 Resource Masters.
Exponential Growth and Decay

**What You’ll Learn**

- Use logarithms to solve problems involving exponential decay.
- Use logarithms to solve problems involving exponential growth.

**Vocabulary**

- rate of decay
- rate of growth

**How can you determine the current value of your car?**

Certain assets, like homes, can appreciate or increase in value over time. Others, like cars, depreciate or decrease in value with time. Suppose you buy a car for $22,000 and the value of the car decreases by 16% each year. The table shows the value of the car each year for up to 5 years after it was purchased.

<table>
<thead>
<tr>
<th>Years after Purchase</th>
<th>Value of Car ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>22,000.00</td>
</tr>
<tr>
<td>1</td>
<td>18,480.00</td>
</tr>
<tr>
<td>2</td>
<td>15,523.20</td>
</tr>
<tr>
<td>3</td>
<td>13,039.49</td>
</tr>
<tr>
<td>4</td>
<td>10,953.17</td>
</tr>
<tr>
<td>5</td>
<td>9200.66</td>
</tr>
</tbody>
</table>

**EXPONENTIAL DECAY** The depreciation of the value of a car is an example of exponential decay. When a quantity decreases by a fixed percent each year, or other period of time, the amount of that quantity after time is given by $y = a(1 - r)^t$, where $a$ is the initial amount and $r$ is the percent of decrease expressed as a decimal. The percent of decrease $r$ is also referred to as the rate of decay.

**Example 1** Exponential Decay of the Form $y = a(1 - r)^t$

**CAFFEINE** A cup of coffee contains 130 milligrams of caffeine. If caffeine is eliminated from the body at a rate of 11% per hour, how long will it take for half of this caffeine to be eliminated from a person’s body?

**Explore** The problem gives the amount of caffeine consumed and the rate at which the caffeine is eliminated. It asks you to find the time it will take for half of the caffeine to be eliminated from a person’s body.

**Plan** Use the formula $y = a(1 - r)^t$. Let $t$ be the number of hours since drinking the coffee. The amount remaining $y$ is half of 130 or 65.

**Solve**

1. $y = a(1 - r)^t$  
   Exponential decay formula
2. $65 = 130(1 - 0.11)^t$  
   Replace $y$ with 65, $a$ with 130, and $r$ with 11% or 0.11.
3. $0.5 = (0.89)^t$  
   Divide each side by 130.
4. $\log 0.5 = \log (0.89)^t$  
   Property of Equality for Logarithms
5. $\log 0.5 = t \log (0.89)$  
   Product Property for Logarithms
6. $\log 0.5 \log 0.89 = t$  
   Divide each side by $\log 0.89$.
7. $5.9480 = t$  
   Use a calculator.

**Study Tip** Rate of Change

Remember to rewrite the rate of change as a decimal before using it in the formula.

**Resource Manager**

**Workbook and Reproducible Masters**

- **Chapter 10 Resource Masters**
  - Study Guide and Intervention, pp. 603–604
  - Skills Practice, p. 605
  - Practice, p. 606
  - Reading to Learn Mathematics, p. 607
  - Enrichment, p. 608
  - Assessment, p. 624

- **Graphing Calculator and Spreadsheet Masters**, p. 46
- **School-to-Career Masters**, p. 20
- **Teaching Algebra With Manipulatives Masters**, p. 278

**Transparencies**

- 5-Minute Check Transparency 10-6
- Answer Key Transparencies

**Technology**

- Interactive Chalkboard
It will take approximately 6 hours for half of the caffeine to be eliminated from a person’s body.

Examine Use the formula to find how much of the original 130 milligrams of caffeine would remain after 6 hours.

\[ y = a(1 - r)^t \quad \text{Exponential decay formula} \]
\[ y = 130(1 - 0.11)^6 \quad \text{Replace } a \text{ with } 130, r \text{ with } 0.11, \text{ and } t \text{ with } 6. \]
\[ y = 64.6 \quad \text{Use a calculator.} \]

Half of 130 is 65, so the answer seems reasonable.

Another model for exponential decay is given by \( y = ae^{-kt} \), where \( k \) is a constant. This is the model preferred by scientists. Use this model to solve problems involving radioactive decay.

**Example 2** Exponential Decay of the Form \( y = ae^{-kt} \)

**PALEONTOLOGY** The half-life of a radioactive substance is the time it takes for half of the atoms of the substance to become disintegrated. All life on Earth contains the radioactive element Carbon-14, which decays continuously at a fixed rate. The half-life of Carbon-14 is 5760 years. That is, every 5760 years half of a mass of Carbon-14 decays away.

a. What is the value of \( k \) for Carbon-14?

To determine the constant \( k \) for Carbon-14, let \( a \) be the initial amount of the substance. The amount \( y \) that remains after 5760 years is then represented by \( \frac{a}{2} \) or 0.5\( a \).

\[ y = ae^{-kt} \quad \text{Exponential decay formula} \]
\[ 0.5a = ae^{-k(5760)} \quad \text{Replace } y \text{ with } 0.5a \text{ and } t \text{ with } 5760. \]
\[ 0.5 = e^{-5760k} \quad \text{Divide each side by } a. \]
\[ \ln 0.5 = \ln e^{-5760k} \quad \text{Property of Equality for Logarithmic Functions} \]
\[ \ln 0.5 = -5760k \quad \text{Inverse Property of Exponents and Logarithms} \]
\[ -\frac{\ln 0.5}{5760} = k \quad \text{Divide each side by } -5760. \]
\[ 0.00012 = k \quad \text{Use a calculator.} \]

The constant for Carbon-14 is 0.00012. Thus, the equation for the decay of Carbon-14 is \( y = ae^{-0.00012t} \), where \( t \) is given in years.

b. A paleontologist examining the bones of a woolly mammoth estimates that they contain only 3% as much Carbon-14 as they would have contained when the animal was alive. How long ago did the mammoth die?

Let \( a \) be the initial amount of Carbon-14 in the animal’s body. Then the amount \( y \) that remains after \( t \) years is 3% of \( a \) or 0.03\( a \).

\[ y = ae^{-0.00012t} \quad \text{Formula for the decay of Carbon-14} \]
\[ 0.03a = ae^{-0.00012t} \quad \text{Replace } y \text{ with } 0.03a. \]
\[ 0.03 = e^{-0.00012t} \quad \text{Divide each side by } a. \]
\[ \ln 0.03 = \ln e^{-0.00012t} \quad \text{Property of Equality for Logarithms} \]
\[ \ln 0.03 = -0.00012t \quad \text{Inverse Property of Exponents and Logarithms} \]
\[ -\frac{\ln 0.03}{0.00012} = t \quad \text{Divide each side by } -0.00012. \]
\[ 29,221 = t \quad \text{Use a calculator.} \]

The mammoth lived about 29,000 years ago.

www.algebra2.com/extra_examples

Teaching Tip In Example 1, point out that you are calculating how long until half the caffeine has been eliminated, which also means half the caffeine remains. If the value to be found is something other than half, students must be careful that they use the formula correctly.
EXPONENTIAL GROWTH

In-Class Examples

3. The population of a city of one million is increasing at a rate of \(3\%\) per year. If the population continues to grow at this rate, in how many years will the population have doubled? D

A 4 years  B 5 years  C 20 years  D 23 years

4. POPULATION As of 2000, Nigeria had an estimated population of 127 million people and the United States had an estimated population of 278 million people. The populations of Nigeria and the United States can be modeled by \(N(t) = 127e^{0.026t}\) and \(U(t) = 278e^{0.009t}\), respectively. According to these models, when will Nigeria’s population be more than the population of the United States?  after 46 years or in 2046

Example 3

Multiple-Choice Test Item

In 1910, the population of a city was 120,000. Since then, the population has increased by exactly \(1.5\%\) per year. If the population continues to grow at this rate, what will the population be in 2010?

A 138,000  B 531,845  C 1,063,690  D \(1.4 \times 10^{11}\)

Read the Test Item:

You need to find the population of the city 2010 – 1910 or 100 years later. Since the population is growing at a fixed percent each year, use the formula \(y = a(1 + r)^t\).

Solve the Test Item:

\[
y = a(1 + r)^t\]

Exponential growth formula

\[
y = 120,000(1 + 0.015)^{100}\]

Replace \(a = 120,000, r = 0.015, \) and \(t = 2010 – 1910 = 100.\)

\[
y = 120,000(1.015)^{100}\]

Simplify.

\[
y = 531,845.48\]

Use a calculator.

The answer is B.

Another model for exponential growth, preferred by scientists, is \(y = ae^{kt}\), where \(k\) is a constant. Use this model to find the constant \(k\).

Example 4

Exponential Growth of the Form \(y = ae^{kt}\)

POPULATION As of 2000, China was the world’s most populous country, with an estimated population of 1.26 billion people. The second most populous country was India, with 1.01 billion. The populations of India and China can be modeled by \(I(t) = 1.01e^{0.015t}\) and \(C(t) = 1.26e^{0.009t}\), respectively. According to these models, when will India’s population be more than China’s?

You want to find \(t\) such that \(I(t) > C(t).\)

\[
I(t) > C(t)
\]

\[
1.01e^{0.015t} > 1.26e^{0.009t}
\]

Replace \(I(t)\) with \(1.01e^{0.015t}\) and \(C(t)\) with \(1.26e^{0.009t}\)

Property of Inequality for Logarithms

\[
\ln 1.01 + 0.015t > \ln 1.26 + 0.009t
\]

Product Property of Logarithms

\[
\ln 1.01 + 0.015t > \ln 1.26 + 0.009t
\]

Property of Inequality for Logarithms

\[
0.006t > 0.006\ln 1.26 - \ln 1.01
\]

Subtract 0.009 from each side.

\[
t > \frac{0.006\ln 1.26 - 0.006\ln 1.01}{0.006}
\]

Divide each side by 0.006.

\[
t > 36.86
\]

Use a calculator.

After 37 years or in 2037, India will be the most populous country in the world.

Standardized Test Practice

Example 3 On all standardized tests, students should look to identify any answer choices that can be logically eliminated. In Example 3, students can quickly determine that since 1% of 120,000 is 1200 and therefore 1.5% is 1800, over the 100 years from 1920 to 2010 the city’s population will have increased by more than 1800(100) or 180,000 people. So Choice A is much too low. Choice D can also be eliminated, because \(1.4 \times 10^{11}\) written in standard notation is 140,000,000,000 (which is 140 billion). That’s more than the population of the entire planet! So, the answer must be either Choice B or Choice C.
Lesson 10-6  Exponential Growth and Decay  563

Check for Understanding

Concept Check
1. Write a general formula for exponential growth and decay where \( r \) is the percent of change. \( y = a(1 + r)^t \), where \( r > 0 \) represents exponential growth and \( r < 0 \) represents exponential decay.
2. Explain how to solve \( y = (1 + r)^t \) for \( t \). See margin. exponential decay
3. OPEN ENDED Give an example of a quantity that grows or decays at a fixed rate. Sample answer: money in a bank

Guided Practice
SPACE For Exercises 4–6, use the following information.
A radioisotope is used as a power source for a satellite. The power output \( P \) (in watts) is given by \( P = 50e^{-0.25t} \), where \( t \) is the time in days.

4. Is the formula for power output an example of exponential growth or decay? Explain your reasoning. Decay; the exponent is negative.
5. Find the power available after 100 days. \( \text{about } 33.5 \text{ watts} \)
6. Ten watts of power are required to operate the equipment in the satellite. How long can the satellite continue to operate? \( \text{about } 402 \text{ days} \)

POPULATION GROWTH For Exercises 7 and 8, use the following information.
The city of Raleigh, North Carolina, grew from a population of 212,000 in 1990 to a population of 259,000 in 1998.

7. Write an exponential growth equation of the form \( y = a e^{kt} \) for Raleigh, where \( t \) is the number of years after 1990. \( y = 212,000 e^{0.025t} \)
8. Use your equation to predict the population of Raleigh in 2010. \( \text{about } 349,529 \text{ people} \)
9. Suppose the weight of a bar of soap decreases by 2.5% each time it is used. If the bar weighs 95 grams when it is new, what is its weight to the nearest gram after 15 uses? \( \text{C} \)
   \[ \begin{array}{|c|c|c|c|c|}
   \hline
   \text{After Uses} & 1 & 2 & 3 & 4 \\
   \text{Weight} & 57.5 \text{ g} & 59.4 \text{ g} & 65 \text{ g} & 93 \text{ g} \\
   \hline
   \end{array} \]

Standardized Test Practice
* indicates increased difficulty

Practice and Apply

Homework Help
For Exercises 10–19 See Examples
10. COMPUTERS Zeus Industries bought a computer for $2500. It is expected to depreciate at a rate of 20% per year. What will the value of the computer be in 2 years? $1600
11. REAL ESTATE The Martins bought a condominium for $85,000. Assuming that the value of the condo will appreciate at most 5% a year, how much will the condo be worth in 5 years? \( \text{at most }$108,484.93 \)
12. MEDICINE Radioactive iodine is used to determine the health of the thyroid gland. It decays according to the equation \( y = ae^{-0.0856t} \), where \( t \) is in days. Find the half-life of this substance. \( \text{about } 8.1 \text{ days} \)
13. PALEONTOLOGY A paleontologist finds a bone that might be a dinosaur bone. In the laboratory, she finds that the Carbon-14 found in the bone is \( \frac{1}{12} \) of that found in living bone tissue. Could this bone have belonged to a dinosaur? Explain your reasoning. (Hint: The dinosaurs lived from 220 million years ago to 63 million years ago.) No; the bone is only about 21,000 years old, and dinosaurs died out 63,000,000 years ago.
14. ANTHROPOLOGY An anthropologist finds there is so little remaining Carbon-14 in a prehistoric bone that instruments cannot measure it. This means that there is less than 0.5% of the amount of Carbon-14 the bones would have contained when the person was alive. How long ago did the person die?

Extra Practice
See page 851.

14. more than 44,000 years ago

Worldwide www.algebra2.com/self_check_quiz
Exponential and Logarithmic Relations

**BIOLOGY**

For Exercises 15 and 16, use the following information.

Bacteria usually reproduce by a process known as binary fission. In this type of reproduction, one bacterium divides, forming two bacteria. Under ideal conditions, some bacteria reproduce every 20 minutes. 15. about 0.0347

15. Find the constant k for this type of bacteria under ideal conditions.

16. Write the equation for modeling the exponential growth of this bacterium.

\[ y = ab^{kt} \]

**ECONOMICS**

For Exercises 17 and 18, use the following information.

The annual Gross Domestic Product (GDP) of a country is the value of all of the goods and services produced in the country during a year. During the period 1985–1999, the Gross Domestic Product of the United States grew about 3.2% per year, measured in 1996 dollars. In 1985, the GDP was $5717 billion.

17. Assuming this rate of growth continues, what will the GDP of the United States be in the year 2010? $12,565 billion

18. In what year will the GDP reach $20 trillion? about 2025

**OLYMPICS**

In 1928, when the high jump was first introduced as a women’s sport at the Olympic Games, the winning women’s jump was 62.5 inches, while the winning men’s jump was 76.5 inches. Since then, the winning jump for women has increased by about 0.38% per year, whereas the winning jump for men has increased at a slower rate, 0.3%. If these rates continue, when will the women’s winning high jump be higher than the men’s? after the year 2182

**HOME OWNERSHIP**

The Mendes family bought a new house 10 years ago for $120,000. The house is now worth $191,000. Assuming a steady rate of growth, what was the yearly rate of appreciation? 4.7%

**WRITING IN MATH**

The half-life of Radium is 1620 years. When will a 20-gram sample of Radium be completely gone? Explain your reasoning.

Never; theoretically, the amount left will always be half of the previous amount.

**Standardized Test Practice**

An artist creates a sculpture out of salt that weighs 2000 pounds. If the sculpture loses 3.5% of its mass each year to erosion, after how many years will the statue weigh less than 1000 pounds? about 19.5 yr

24. The curve shown at the right represents a portion of the graph of which function? D

- **A** \[ y = 50 - x \]
- **B** \[ y = e^{-x} \]
- **D** \[ y = e^{x} \]
- **E** \[ y = \log x \]

xy = 5

O

x

564 Chapter 10 Exponential and Logarithmic Relations

**Study Guide and Intervention, p. 605 (shown) and p. 604**

Exponential Decay

The exponential decay formula is an example of exponential decay. When a quantity decreases at a fixed percent each time period, the amount of the quantity after t time periods is given by \( y = a(1 - r)^t \), where \( a \) is the initial amount and \( -r \) is the percent decrease expressed as a decimal.

**CONSUMER PRICES**

An exponential growth function is often used by economists to forecast the future. Knowledge of the rate of change helps businesses make decisions about the quantity of goods they will produce or purchase.

**LO**

The amount of the quantity after exponential decay

Depreciation of value and radioactive decay are examples of exponential decay.

**RADIOACTIVE DECAY**

Visualizing their graphs is often a good way to remember the difference between exponential growth and decay.

- **a.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **b.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **c.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **d.** If its price decreases by 6% per year, how much will it be worth in 5 years?

**BUSINESS**

- **a.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **b.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **c.** If its price decreases by 6% per year, how much will it be worth in 5 years?
- **d.** If its price decreases by 6% per year, how much will it be worth in 5 years?

**Exercises**

State whether each equation represents exponential growth or decay.

For Exercises 29 and 30, use the following information.

- **29.** Which equation represents the exponential growth of this bacterium?
- **30.** Which equation represents the exponential decay of this bacterium?

**SAMPLE ANSWER:** If \( \frac{d}{dt} > 0 \), the graph of \( y = ae^{kt} \) is always increasing if \( k > 0 \) and is always decreasing if \( k < 0 \). Since \( k \) is always a positive number, if \( \frac{d}{dt} > 0 \) the base will be greater than 1 and the function will be increasing. Conversely, if the base will be less than 1 and the function will be decreasing.

**More About...**

**Exponential and Logarithmic Relations**

**Standardized Test Practice**

Effective Annual Yield

When interest is compounded more than once per year, the effective annual yield is higher than the nominal interest rate. The effective annual yield \( r_{eff} \) is the interest rate that would give the same amount of interest if the interest were compounded per year. The formula for the effective annual yield is:

\[ r_{eff} = \left(1 + \frac{r}{n}\right)^n - 1 \]

where \( r \) is the nominal interest rate and \( n \) is the number of times interest is compounded per year.

**Enrichment, p. 608**

When interest is compounded more than once per year, the effective annual yield is higher than the nominal interest rate. The effective annual yield \( r_{eff} \) is the interest rate that would give the same amount of interest if the interest were compounded per year. The formula for the effective annual yield is:

\[ r_{eff} = \left(1 + \frac{r}{n}\right)^n - 1 \]

where \( r \) is the nominal interest rate and \( n \) is the number of times interest is compounded per year.
Maintain Your Skills

Write an equivalent exponential or logarithmic equation. (Lesson 10-5)

25. $e^y = y$  $\ln y = 3$
26. $e^{\ln x - 2} = 29$
27. $\ln 4 + 2 \ln x = 8$

Solve each equation or inequality. Round to four decimal places. (Lesson 10-4)

28. $16^x = 70$  1.5323
29. $2^{3p} > 1000$  $p > 3.3219$
30. $\log_8 81 = 2.9$

BUSINESS  For Exercises 31–33, use the following information.

The board of a small corporation decided that 8% of the annual profits would be divided among the six managers of the corporation. There are two sales managers and four nonsales managers. Fifty percent of the amount would be split equally among all six managers. The other 50% would be split among the four nonsales managers. Let $p$ represent the annual profits of the corporation. (Lesson 9-2)

31. Write an expression to represent the share of the profits each nonsales manager will receive.

32. Simplify this expression. $\frac{0.5(0.08p)}{6} + \frac{0.5(0.08p)}{4}$

33. Write an expression in simplest form to represent the share of the profits each sales manager will receive. $\frac{p}{150}$

Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. (Lesson 8-6)

34. $4y^2 - 3x^2 + 8y - 24x = 50$ hyperbola
35. $7x^2 - 42x + 6y^2 - 24y = -45$ ellipse
36. $y^2 + 3x - 8y = 4$ parabola
37. $x^2 + y^2 - 6x + 2y + 5 = 0$ circle

AGRICULTURE  For Exercises 38–40, use the graph at the right. (Lesson 5-1)

38. Write the number of pounds of pecans produced by U.S. growers in 2000 in scientific notation. $2.06 \times 10^8$

39. Write the number of pounds of pecans produced by the state of Georgia in 2000 in scientific notation. $8 \times 10^7$

40. What percent of the overall pecan production for 2000 can be attributed to Georgia? about 38.8%

USA TODAY Snapshots®

Georgia led pecan production in 2000

U.S. growers produced more than 206 million pounds of pecans in 2000. States producing the most pecans (in pounds):

- Georgia: 80 million
- New Mexico: 32 million
- Texas: 30 million
- Louisiana: 17 million
- Alabama: 15 million
- Arizona: 14 million

Source: National Agricultural Statistics Service

By Sam Ward, USA TODAY

On Quake Anniversary, Japan Still Worries

It is time to complete your project. Use the information and data you have gathered about earthquakes to prepare a research report or Web page. Be sure to include graphs, tables, diagrams, and any calculations you need for the earthquake you chose.

www.algebra2.com/webquest

Lesson 10-6  Exponential Growth and Decay 565

Online Lesson Plans

USA TODAY Education’s Online site offers resources and interactive features connected to each day’s newspaper. Experience TODAY, USA TODAY’s daily lesson plan, is available on the site and delivered daily to subscribers. This plan provides instruction for integrating USA TODAY graphics and key editorial features into your mathematics classroom. Log on to www.education.usatoday.com.
Vocabulary and Concept Check

- This alphabetical list of vocabulary terms in Chapter 10 includes a page reference where each term was introduced.
- Assessment: A vocabulary test/review for Chapter 10 is available on p. 622 of the Chapter 10 Resource Masters.

Lesson-by-Lesson Review

For each lesson,
- the main ideas are summarized,
- additional examples review concepts, and
- practice exercises are provided.

Vocabulary PuzzleMaker

The Vocabulary PuzzleMaker software improves students’ mathematics vocabulary using four puzzle formats—crossword, scramble, word search using a word list, and word search using clues. Students can work on a computer screen or from a printed handout.

MindJogger Videoquizzes

MindJogger Videoquizzes provide an alternative review of concepts presented in this chapter. Students work in teams in a game show format to gain points for correct answers. The questions are presented in three rounds.

Round 1 Concepts (5 questions)
Round 2 Skills (4 questions)
Round 3 Problem Solving (4 questions)

Vocabulary and Concept Check

- Change of Base Formula (p. 548)
- common logarithm (p. 547)
- exponential decay (p. 524)
- exponential equation (p. 526)
- exponential function (p. 524)
- exponential growth (p. 524)
- exponential inequality (p. 527)
- logarithm (p. 531)
- logarithmic equation (p. 533)
- logarithmic function (p. 532)
- logarithmic inequality (p. 533)
- natural base, e (p. 554)
- natural base exponential function (p. 554)
- natural logarithm (p. 554)
- natural logarithmic function (p. 554)
- Power Property of Logarithms (p. 543)
- Product Property of Logarithms (p. 541)
- Property of Equality for Exponential Functions (p. 526)
- Property of Equality for Logarithmic Functions (p. 534)
- Property of Inequality for Exponential Functions (p. 527)
- Property of Inequality for Logarithmic Functions (p. 534)
- Quotient Property of Logarithms (p. 542)
- rate of decay (p. 560)
- rate of growth (p. 562)

State whether each sentence is true or false. If false, replace the underlined word(s) to make a true statement.

1. If $24^{y} + 3 = 24^{y} - 4$, then $2y + 3 = y - 4$ by the **Property of Equality for Exponential Functions**. 
2. The number of bacteria in a petri dish over time is an example of **exponential decay**. 
3. The **natural logarithm** is the inverse of the exponential function with base 10. 
4. The **Power Property of Logarithms** shows that $\ln 9 < \ln 81$. 
5. If a savings account yields 2% interest per year, then 2% is the **rate of growth**. 
6. Radioactive half-life is an example of **exponential decay**. 
7. The inverse of an exponential function is a **composite function**. 
8. The **Quotient Property of Logarithms** is shown by $\log_{4} \frac{2x}{2 + \log_{4} x}$. 
9. The function $f(x) = 2(5)^{x}$ is an example of a **quadratic function**. 

Lesson-by-Lesson Review

**Exponential Functions**

**Concept Summary**

- An exponential function is in the form $y = ab^{x}$, where $a \neq 0$, $b > 0$, and $b \neq 1$.
- The function $y = ab^{x}$ represents exponential growth for $a > 0$ and $b > 1$, and exponential decay for $a > 0$ and $0 < b < 1$.
- Property of Equality for Exponential Functions: If $b$ is a positive number other than 1, then $b^{x} = b^{y}$ if and only if $x = y$.
- Property of Inequality for Exponential Functions: If $b > 1$, then $b^{x} > b^{y}$ if and only if $x > y$, and $b^{x} < b^{y}$ if and only if $x < y$.

For more information about Foldables, see Teaching Mathematics with Foldables.

Have students look through the chapter to make sure they have included notes and examples for each lesson in this chapter in their Foldable.

Encourage students to refer to their Foldables while completing the Study Guide and Review and to use them in preparing for the Chapter Test.
**Example**

Solve $64 = 2^{3n + 1}$ for $n$.

1. $64 = 2^{3n + 1}$ Original equation
2. $2^6 = 2^{3n + 1}$ Rewrite 64 as $2^6$ so each side has the same base.
3. $6 = 3n + 1$ Property of Equality for Exponential Functions
4. $\frac{5}{3} = n$ The solution is $\frac{5}{3}$.

**Exercises**

Determine whether each function represents exponential growth or decay. See Example 2 on page 525.

10. $y = 5(0.7)^x$ decay
11. $y = \frac{1}{3}(4)^x$ growth

Write an exponential function whose graph passes through the given points. See Example 3 on page 525.

12. $(0, -2)$ and $(3, -54)$ $y = -2(3)^x$
13. $(0, 7)$ and $(1, 1.4)$ $y = 7(\frac{1}{5})^x$

Solve each equation or inequality. See Examples 5 and 6 on pages 526 and 527.

14. $9^x = \frac{1}{81}$ $-2$
15. $2^6x = 4^{5x + 2}$ $1$
16. $49^{3p + 1} = 7^{2p - 5}$ $\frac{7}{4}$
17. $9^{x^2} \leq 27^{x^2 - 2}$ $x \leq -\sqrt{6}$ or $x \geq \sqrt{6}$

**Logarithms and Logarithmic Functions**

**Concept Summary**

- Suppose $b > 0$ and $b \neq 1$. For $x > 0$, there is a number $y$ such that $\log_b x = y$ if and only if $b^y = x$.
- Logarithmic to exponential inequality:
  - If $b > 1$, $x > 0$, and $\log_b x < y$, then $x > b^y$.
  - If $b > 1$, $x > 0$, and $\log_b x > y$, then $0 < x < b^y$.
- Property of Equality for Logarithmic Functions:
  - If $b$ is a positive number other than 1, then $\log_b x = \log_b y$ if and only if $x = y$.
- Property of Inequality for Logarithmic Functions:
  - If $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

**Examples**

1. Solve $\log_9 n > \frac{3}{2}$

- $\log_9 n > \frac{3}{2}$ Original inequality
- $n > 9^{\frac{3}{2}}$ Logarithmic to exponential inequality
- $n > (3^2)^{\frac{1}{2}}$ $9 = 3^2$
- $n > 3^3$ Power of a Power
- $n > 27$ Simplify.
2. Solve \( \log_3 12 = \log_3 2x \).

\[
\begin{align*}
\log_3 12 &= \log_3 2x & \text{Original equation} \\
12 &= 2x & \text{Property of Equality for Logarithmic Functions} \\
6 &= x & \text{Divide each side by 2.}
\end{align*}
\]

**Exercises** Write each equation in logarithmic form. See Example 1 on page 532.

18. \( 7^3 = 343 \) \( \log_7 343 = 3 \)

19. \( 5^{-2} = \frac{1}{25} \) \( \log_5 \frac{1}{25} = -2 \)

20. \( 4^{\frac{1}{2}} = 2 \) \( \log_4 2 = \frac{1}{2} \)

**Write each equation in exponential form.** See Example 2 on page 532.

21. \( \log_4 64 = 3 \) \( 4^3 = 64 \)

22. \( \log_8 2 = \frac{1}{3} \) \( 8^{\frac{1}{3}} = 2 \)

23. \( \log_6 \frac{1}{36} = -2 \)

**Evaluate each expression.** See Examples 3 and 4 on pages 532 and 533.

24. \( 4^{\log_4 9} = 9 \)

25. \( \log_7 7^{-5} = -5 \)

26. \( \log_8 3^{\frac{1}{4}} = \frac{1}{4} \)

27. \( \log_{13} 169 = 2 \)

**Solve each equation or inequality.** See Examples 5–8 on pages 533 and 534.

28. \( \log_4 x = \frac{1}{2} \) \( 2 \)

29. \( \log_{81} 729 = x \) \( \frac{3}{2} \)

30. \( \log_9 2 = 3 \)

31. \( \log_8 (3y - 1) < \log_8 (y + 5) \) \( \frac{1}{3} < y < 3 \)

32. \( \log_5 12 < \log_5 (5x - 3) \) \( x > 3 \)

33. \( \log_6 (x^2 + x) = \log_6 12 - 4, 3 \)

### Properties of Logarithms

**Concept Summary**

- The logarithm of a product is the sum of the logarithms of its factors.
- The logarithm of a quotient is the difference of the logarithms of the numerator and the denominator.
- The logarithm of a power is the product of the logarithm and the exponent.

**Example**

Use \( \log_{12} 9 = 0.884 \) and \( \log_{12} 18 = 1.163 \) to approximate the value of \( \log_{12} 2 \).

\[
\begin{align*}
\log_{12} 2 &= \log_{12} \frac{18}{9} & \text{Replace 2 with } \frac{18}{9}. \\
&= \log_{12} 18 - \log_{12} 9 & \text{Quotient Property} \\
&= 1.163 - 0.884 \text{ or } 0.279 & \text{Replace } \log_{12} 9 \text{ with } 0.884 \text{ and } \log_{12} 18 \text{ with } 1.163.
\end{align*}
\]

**Exercises** Use \( \log_7 7 = 0.8856 \) and \( \log_8 4 = 0.6309 \) to approximate the value of each expression. See Examples 1 and 2 on page 542.

34. \( \log_9 28 \) \( 1.5165 \)

35. \( \log_9 49 \) \( 1.7712 \)

36. \( \log_9 144 \) \( 2.2618 \)

**Solve each equation.** See Example 5 on page 543.

37. \( \log_2 y = \frac{1}{3} \log_2 27 \) \( 3 \)

38. \( \log_5 7 + \frac{1}{2} \log_5 4 = \log_5 x \) \( 14 \)

39. \( 2 \log_2 x - \log_2 (x + 3) = 2 \) \( 6 \)

40. \( \log_3 x - \log_3 4 = \log_3 12 \) \( 48 \)

41. \( \log_6 48 - \log_6 \frac{16}{5} + \log_6 5 = \log_6 5x \) \( 15 \)

42. \( \log_7 m = \frac{1}{3} \log_7 64 + \frac{1}{2} \log_7 121 \) \( 44 \)
**Common Logarithms**

**Concept Summary**
- Base 10 logarithms are called common logarithms and are usually written without the subscript 10: \( \log_{10} x \).
- You use the inverse of logarithms, or exponentiation, to solve equations or inequalities involving common logarithms: \( 10^{\log_{10} x} = x \).
- The Change of Base Formula: \( \log_{b} n = \frac{\log_{a} n}{\log_{a} b} \)← log base \( b \) original number ← log base \( b \) old base.

**Example**

Solve \( 5^x = 7 \).

\[
5^x = 7 \quad \text{Original equation}
\]

\[
\log 5^x = \log 7 \quad \text{Property of Equality for Logarithmic Functions}
\]

\[
x \log 5 = \log 7 \quad \text{Power Property of Logarithms}
\]

\[
x = \frac{\log 7}{\log 5} \quad \text{Divide each side by log 5.}
\]

\[
x = 0.8451 \text{ or } 1.2090 \quad \text{Use a calculator.}
\]

**Exercises**

Solve each equation or inequality. Round to four decimal places.

See Examples 3 and 4 on page 548.

43. \( 2^x = 53 \) \( 5.7279 \)
44. \( 2.3^x = 66.6 \) \( \pm 2.2452 \)
45. \( 3^x - 7 < 4^x + 3 \) \( x < 7.3059 \)
46. \( 6^y = 8^y - 1 \) \( -0.6309 \)
47. \( 12^y - 5 \geq 9.32 \) \( x \geq 5.8983 \)
48. \( 2.1^x - 5 = 9.32 \) \( 8.0086 \)

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places.

See Example 5 on page 549.

49. \( \log_{20} 1000 \) \( \log_{20} 1000 = 2.3059 \)
50. \( \log_{2} 15 \) \( \log_{2} 15 = 3.9069 \)
51. \( \log_{10} 1000 \) \( \log_{10} 1000 = 3.0000 \)

---

**Base e and Natural Logarithms**

**Concept Summary**
- You can write an equivalent base \( e \) exponential equation for a natural logarithmic equation and vice versa by using the fact that \( \ln x = \log_{e} x \).
- Since the natural base function and the natural logarithmic function are inverses, these two functions can be used to “undo” each other.
- \( e^{\ln x} = x \) and \( \ln e^x = x \).

**Example**

Solve \( \ln (x + 4) > 5 \).

\[
\ln (x + 4) > 5 \quad \text{Original inequality}
\]

\[
e^{\ln (x + 4)} > e^5 \quad \text{Write each side using exponents and base } e.
\]

\[
x + 4 > e^5 \quad \text{Inverse Property of Exponents and Logarithms}
\]

\[
x > e^5 - 4 \quad \text{Subtract 4 from each side.}
\]

\[
x > 144.4132 \quad \text{Use a calculator.}
\]

---

**Chapter 10 Study Guide and Review**
Exercises  Write an equivalent exponential or logarithmic equation.  
See Example 3 on page 555.

52.  \( e^x = 6 \quad \ln 6 = x \)  
53.  \( 7.4 = x \quad e^x = 7.4 \)

Evaluate each expression.  See Example 4 on page 555.

54.  \( e^{\ln 12} = 12 \)  
55.  \( \ln e^x = 7x \)

Solve each equation or inequality.  
See Examples 5 and 7 on pages 555 and 556.

56.  \( 2e^x = 4 \quad 10.9163 \)  
57.  \( e^{2x} = 3.2 \quad x > 1.1632 \)  
58.  \( -4e^{2x} + 15 = 7 \quad 0.3466 \)

59.  \( \ln 3x \leq 5 \)  
60.  \( \ln (x - 10) = 0.5 \quad 11.5487 \)  
61.  \( \ln x + \ln 4x = 10 \quad 74.2066 \)

Exponential Growth and Decay

Concept Summary

- Exponential decay: \( y = a(1 - r)^t \) or \( y = ae^{-kt} \)
- Exponential growth: \( y = a(1 + r)^t \) or \( y = ae^{kt} \)

Example  
BIOLOGY  A certain culture of bacteria will grow from 500 to 4000 bacteria in 1.5 hours. Find the constant \( k \) for the growth formula. Use \( y = ne^{kt} \).

\[
4000 = 500e^{k(1.5)}
\]

Replace \( y \) with 4000, \( a \) with 500, and \( t \) with 1.5.

\[
8 = e^{1.5k}
\]

Divide each side by 500.

\[
\ln 8 = \ln e^{1.5k}
\]

Property of Equality for Logarithmic Functions

\[
\ln 8 = 1.5k
\]

Inverse Property of Exponents and Logarithms

\[
\frac{\ln 8}{1.5} = k
\]

Divide each side by 1.5.

\[
1.3863 = k
\]

Use a calculator.

Exercises  See Examples 1–4 on pages 560–562.

62. BUSINESS  Able Industries bought a fax machine for $250. It is expected to depreciate at a rate of 25% per year. What will be the value of the fax machine in 3 years? \( \$105.47 \)

63. BIOLOGY  For a certain strain of bacteria, \( k \) is 0.872 when \( t \) is measured in days. How long will it take 9 bacteria to increase to 738 bacteria? \( 5.05 \) days

64. CHEMISTRY  Radium-226 decomposes radioactively. Its half-life, the time it takes for half of the sample to decompose, is 1800 years. Find the constant \( k \) in the decay formula for this compound. \( \text{about} -0.000385 \)

65. POPULATION  The population of a city 10 years ago was 45,600. Since then, the population has increased at a steady rate each year. If the population is currently 64,800, find the annual rate of growth for this city. \( \text{about} 3.6\% \)
Vocabulary and Concepts

Choose the term that best completes each sentence.
1. The equation \( y = 0.3(4)^x \) is an exponential (growth, decay) function.
2. The logarithm of a quotient is the (sum, difference) of the logarithms of the numerator and the denominator.
3. The base of a natural logarithm is \((10, e)\).

Skills and Applications

4. Write \( 3^7 = 2187 \) in logarithmic form. \( \log_3 2187 = 7 \)
5. Write \( \log_8 16 = \frac{4}{3} \) in exponential form. \( 8^{\frac{4}{3}} = 16 \)
6. Write an exponential function whose graph passes through \((0, 0.4)\) and \((2, 6.4)\). \( y = 0.4(4)^x \)
7. Express \( \log_3 5 \) in terms of common logarithms.
8. Evaluate \( \log_2 \frac{1}{32} = -5 \)

Use \( \log_4 7 \approx 1.4037 \) and \( \log_4 3 \approx 0.7925 \) to approximate the value of each expression.
9. \( \log_4 21 \approx 2.1962 \)
10. \( \log_4 \frac{7}{12} \approx -0.3888 \)

Simplify each expression.
11. \( \sqrt[3]{8} + \sqrt{2} \approx 1.81 
12. \( 81^{\frac{1}{3}} + 3^{\frac{1}{2}} \cdot 3^{\frac{3}{2}} = 3^{\frac{3}{2}} + 3^\frac{3}{2} = 3 \cdot \sqrt{3} + 3 \cdot \sqrt{3} = 6 \cdot \sqrt{3} \)

Solve each equation or inequality. Round to four decimal places if necessary.
13. \( 2^x - 3 = \frac{1}{16} \rightarrow x = 1 \)
14. \( 2^{2x} + 1 = 3^{y} - 1 \rightarrow x = 0.5 \)
15. \( \log_2 x < 7 \rightarrow 0 < x < 128 \)
16. \( \log_5 0.4(4)^x \)
17. \( \log_5 x = 3 \rightarrow x = 5 \)
18. \( \log_9 (x + 4) + \log_9 (x - 4) = 1 \rightarrow \)
19. \( \log_8 (8y - 7) = \log_8 (9y + 5) \rightarrow 4 \)
20. \( \log_3 \left( \frac{3^{4x - 1}}{15} \right) = 4 \rightarrow 3^{4x - 1} = 15 \)
21. \( 7.6 - 1 = 43 \rightarrow 3.9910 \)
22. \( \log_5 2 + \frac{3}{5} \log_2 27 = \log_5 x \rightarrow 3.1507 \)
23. \( 3^x = 5^{x-1} \rightarrow 3.5 \)
24. \( 4^{x + 3} = 9^{x + 3} \rightarrow 18.6848 \)
25. \( e^{3y} > 6 \rightarrow y > \log_e \left( \frac{6}{e} \right) \approx 0.9730 \)
26. \( 2e^{3x} + 5 = 11 \rightarrow 0.3662 \)
27. \( \ln 3x - \ln 15 = 2 \rightarrow 3.6953 \)

COINS For Exercises 28 and 29, use the following information.
You buy a commemorative coin for $25. The value of the coin increases 3.25% per year.
28. How much will the coin be worth in 15 years? $40.39
29. After how many years will the coin have doubled in value? 22

30. QUANTITATIVE COMPARISON Compare the quantity in Column A and the quantity in Column B. Then determine whether:
(A) the quantity in Column A is greater,
(B) the quantity in Column B is greater,
(C) the two quantities are equal, or
(D) the relationship cannot be determined

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100 was deposited in an account 5 years ago.</td>
<td>the current value of the account if the annual interest rate is 3% compounded quarterly</td>
</tr>
<tr>
<td>the current value of the account if the annual interest rate is 3% compounded continuously</td>
<td></td>
</tr>
</tbody>
</table>

Portfolio Suggestion

Introduction In mathematics, exponential functions can be used to model real-world problems. The solution to the exponential function provides a solution to the real-world problem.

Ask Students Find a real-world problem modeled by an exponential function from your work in this chapter and show how you solved it. Explain how the function models the real-world situation and what could be gained by understanding the real-world problem better. Place your work in your portfolio.
Chapter 10  
Standardized Test Practice

These two pages contain practice questions in the various formats that can be found on the most frequently given standardized tests.

A practice answer sheet for these two pages can be found on p. A1 of the Chapter 10 Resource Masters.

Additional Practice
See pp. 627–628 in the Chapter 10 Resource Masters for additional standardized test practice.

Part 1  Multiple Choice

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

1. The arc shown is part of a circle. Find the area of the shaded region. B
   - A) 8π units²
   - B) 16π units²
   - C) 32π units²
   - D) 64π units²

2. If line ℓ is parallel to line m in the figure below, what is the value of x? D
   - A) 40
   - B) 50
   - C) 60
   - D) 70

3. According to the graph, what was the percent of increase in sales from 1998 to 2000? D
   - A) 5%
   - B) 15%
   - C) 25%
   - D) 50%

4. What is the x-intercept of the line described by the equation y = 2x + 5? B
   - A) −5
   - B) −5/2
   - C) 0
   - D) 5/2

5. \( \frac{(xy)^2 - 4}{y^2 + x^3} = \) D
   - A) \( \frac{1}{x^2y} \)
   - B) \( \frac{x}{z} \)
   - C) \( \frac{z}{x} \)
   - D) \( \frac{1}{x} \)

6. If \( 6 - v = 10 \), then v = A
   - A) −16
   - B) −4
   - C) 4
   - D) 8

7. The expression \( \frac{1}{3}\sqrt{45} \) is equivalent to A
   - A) \( \sqrt{5} \)
   - B) \( 3\sqrt{5} \)
   - C) \( 5 \)
   - D) \( 15 \)

8. What are all the values for x such that \( x^2 \leq 3x + 18 \)? B
   - A) \( x < -3 \)
   - B) \( -3 < x < 6 \)
   - C) \( x > -3 \)
   - D) \( x < 6 \)

9. If \( f(x) = 2x^3 - 18x \), what are all the values of x at which \( f(x) = 0 \)? B
   - A) 0, 3
   - B) −3, 0, 3
   - C) −6, 0, 6
   - D) −3, 2, 3

10. Which of the following is equal to \( \frac{17.5(10^{-2})}{500(10^{-4})} \)? D
    - A) 0.035(10⁻²)
    - B) 0.35(10⁻²)
    - C) 0.0035(10²)
    - D) 0.035(10²)

Test-Taking Tip

Question 7  You can use estimates to help you eliminate answer choices. For example, in Question 7, you can estimate that \( \frac{1}{3}\sqrt{45} \) is less than \( \frac{1}{3}\sqrt{49} \), which is \( \frac{1}{3} \) or \( 2\frac{1}{3} \). Eliminate choices C and D.

Log On for Test Practice
The Princeton Review offers additional test-taking tips and practice problems at their web site. Visit www.princetonreview.com or www.review.com

TestCheck and Worksheet Builder
Special banks of standardized test questions similar to those on the SAT, ACT, TIMSS 8, NAEP 8, and Algebra 1 End-of-Course tests can be found on this CD-ROM.
Part 2 Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

11. If the outer diameter of a cylindrical tank is 62.46 centimeters and the inner diameter is 53.32 centimeters, what is the thickness of the tank? $4.57$ cm

12. What number added to 80% of itself is equal to 45? $25$

13. Of 200 families surveyed, 95% have at least one TV and 60% of those with TVs have more than 2 TVs. If 50 families have exactly 2 TVs, how many families have exactly 1 TV? $26$

14. In the figure, if $ED = 8$, what is the measure of line segment $AE$? $2$

15. If $a \rightarrow b$ is defined as $a - b + ab$, find the value of $4 \rightarrow 2$. $10$

16. If $6(m + k) = 26 + 4(m + k)$, what is the value of $m + k$? $13$

17. If $y = 1 - x^2$ and $-3 \leq x \leq 1$, what number is found by subtracting the least possible value of $y$ from the greatest possible value of $y$? $9$

18. If $f(x) = (x - \pi)(x - 3)(x - e)$, what is the difference between the greatest and least roots of $f(x)$? Round to the nearest hundredth. $42$

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Part 3 Quantitative Comparison

Compare the quantity in Column A and the quantity in Column B. Then determine whether:

A the quantity in Column A is greater,
B the quantity in Column B is greater,
C the two quantities are equal, or
D the relationship cannot be determined from the information given.

19. $-1 < xy < 0$

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
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<tbody>
<tr>
<td>$x + y$</td>
<td>$xy$</td>
</tr>
</tbody>
</table>

20. $z = x + y$

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$\sqrt{r^2 + s^2}$</td>
</tr>
</tbody>
</table>

21. $x - y + z = 5$

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + z$</td>
<td></td>
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</table>

22. $x - y + z = 9$

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2nx$</td>
<td>$(x - n)^2$</td>
</tr>
</tbody>
</table>

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Page 521, Chapter 10 Getting Started

9. ![Graph 9](image1.png)
10. ![Graph 10](image2.png)

11. ![Graph 11](image3.png)
12. ![Graph 12](image4.png)

Page 522, Preview of Lesson 10-1

Algebra Activity

3. Sample graph:

![Sample Graph](image5.png)

Page 524, Lesson 10-1

Graphing Calculator Investigation

4a. As the value of \( x \) increases, the value of \( y \) for the graph of \( y = 4^x \) increases faster than for the graph of \( y = 3^x \), and the value of \( y \) for the graph of \( y = 3^x \) increases faster than for the graph of \( y = 2^x \). The graphs have the same domain, all real numbers, and range, \( y > 0 \). They have the same asymptote, the \( x \)-axis, and the same \( y \)-intercept, 1.

4b. As the value of \( x \) increases, the value of \( y \) for the graph of \( y = (\frac{1}{3})^x \) decreases faster than for the graph of \( y = (\frac{1}{2})^x \), and the value of \( y \) for the graph of \( y = (\frac{1}{4})^x \) decreases faster than for the graph of \( y = (\frac{1}{5})^x \). The graphs have the same domain, all real numbers, and range, \( y > 0 \). They have the same asymptote, the \( x \)-axis, and the same \( y \)-intercept, 1.

4c. The graph of \( y = 3(2)^x \) moves down and to the right more quickly than the graph of \( y = -1(2)^x \). The graph of \( y = 3(2)^x \) moves up and to the right more quickly than the graph of \( y = 2^x \). All of the graphs have the same domain, all real numbers, and asymptote, the \( x \)-axis, but the range of \( y = -3(2)^x \) and \( y = -1(2)^x \) is \( y < 0 \), while the range of \( y = 2^x \) and \( y = 3(2)^x \) is \( y > 0 \). The \( y \)-intercept of \( y = -3(2)^x \) is \(-3\), of \( y = -1(2)^x \) is \(-1\), of \( y = 2^x \) is 1, and of \( y = 3(2)^x \) is 3.

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21. \( D = \{ x \} | x \text{ is all real numbers.} \),
\( R = \{ y \} | y > 0 \)

![Graph 21](image6.png)

22. \( D = \{ x \} | x \text{ is all real numbers.} \),
\( R = \{ y \} | y > 0 \)

![Graph 22](image7.png)

23. \( D = \{ x \} | x \text{ is all real numbers.} \),
\( R = \{ y \} | y > 0 \)

![Graph 23](image8.png)

24. \( D = \{ x \} | x \text{ is all real numbers.} \),
\( R = \{ y \} | y > 0 \)

![Graph 24](image9.png)
68. The number of teams \( y \) that could compete in a tournament in which \( x \) rounds are played can be expressed as \( y = 2^x \). The 2 teams that make it to the final round got there as a result of winning games played with 2 other teams, for a total of \( 2 \times 2 = 2^2 \) or 4 games played in the previous round or semifinal round. Answers should include the following.

- Rewrite 128 as a power of 2, \( 2^7 \). Substitute \( 2^7 \) for \( y \) in the equation \( y = 2^x \). Then, using the Property of Equality for Exponents, \( x \) must be 7. Therefore, 128 teams would need to play 7 rounds of tournament play.
- Sample answer: 52 would be an inappropriate number of teams to play in this type of tournament because 52 is not a power of 2.

71. The graphs have the same shape. The graph of \( y = 2^x + 3 \) is the graph of \( y = 2^x \) translated three units up. The asymptote for the graph of \( y = 2^x \) is the line \( y = 0 \) and for \( y = 2^x + 3 \) is the line \( y = 3 \). The graphs have the same domain, all real numbers, but the range of \( y = 2^x \) is \( y > 0 \) and the range of \( y = 2^x + 3 \) is \( y > 3 \). The \( y \)-intercept of the graph of \( y = 2^x \) is 1 and for the graph of \( y = 2^x + 3 \) is 4.

72. The graphs have the same shape. The graph of \( y = 3^{x + 1} \) is the graph of \( y = 3^x \) translated one unit to the left. The asymptote for the graph of \( y = 3^x \) and for \( y = 3^{x + 1} \) is the line \( y = 0 \). The graphs have the same domain, all real numbers, and range, \( y > 0 \). The \( y \)-intercept of the graph of \( y = 3^x \) is 1 and for the graph of \( y = 3^{x + 1} \) is 3.
73. A logarithmic scale illustrates that values next to each other vary by a factor of 10. Answers should include the following.

- Pin drop: $1 \times 10^0$; Whisper: $1 \times 10^2$; Normal conversation: $1 \times 10^6$; Kitchen noise: $1 \times 10^{10}$; Jet engine: $1 \times 10^{12}$

- On the scale shown above, the sound of a pin drop and the sound of normal conversation appear not to differ by much at all, when in fact they do differ in terms of the loudness we perceive. The first scale shows this difference more clearly.

48. Since logarithms are exponents, the properties of logarithms are similar to the properties of exponents. The Product Property states that to multiply two powers that have the same base, add the exponents. Similarly, the logarithm of a product is the sum of the logarithms of its factors. The Quotient Property states that to divide two powers that have the same base, subtract their exponents. Similarly, the logarithm of a quotient is the difference of the logarithms of the numerator and the denominator. The Power Property states that to find the power of a power, multiply the exponents. Similarly, the logarithm of a power is the product of the logarithm and the exponent. Answers should include the following.

- **Quotient Property:** \( \log_b \left( \frac{32}{8} \right) = \log_b \left( \frac{2^5}{2^3} \right) \)
  \[
  = \log_b 2^2 = \log_b 4 \quad \text{Quotient of Powers}
  \]
  \[
  = \log_b 2^5 - \log_b 2^3 \quad \text{Inverse Property of Exponents and Logarithms}
  \]
  \[
  \log_b 32 - \log_b 8 = \log_b 2^5 - \log_b 2^3 \quad \text{Replace 32 with } 2^5 \text{ and } 8 \text{ with } 2^3.
  \]
  \[
  = 5 - 3 \text{ or } 2 \quad \text{Inverse Property of Exponents and Logarithms}
  \]
  So, \( \log_b \left( \frac{32}{8} \right) = \log_b 32 - \log_b 8 \).

- **Power Property:** \( \log_3 9^4 = \log_3 (3^2)^4 \)
  \[
  = \log_3 3^{2 \cdot 4} = 2 \cdot 4 \quad \text{Power of a Power}
  \]
  \[
  = 8 \quad \text{Inverse Property of Exponents and Logarithms}
  \]
  So, \( \log_3 9^4 = 4 \log_3 9 \).

- The Product of Powers Property and Product Property of Logarithms both involve the addition of exponents, since logarithms are exponents.

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62. \[
\frac{\log x}{\log y} = \frac{\ln x}{\ln y} \quad \text{Original statement}
\]
\[
\log x = \frac{\log x}{\log e} \quad \text{Change of Base Formula}
\]
\[
\log x = \log x \cdot \log e \quad \text{Multiply } \frac{\log x}{\log e} \text{ by the reciprocal of } \frac{\log y}{\log e}
\]
\[
\log x \cdot \log y = \frac{\log x}{\log y} \quad \text{Simplify.}
\]